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Kasac, Josip; Kotarski, Denis; Piljek, Petar

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FREQUENCY-SHIFTING-BASED ALGEBRAIC APPROACH TO STABLE ON-LINE PARAMETER IDENTIFICATION AND STATE ESTIMATION OF MULTIROTOR UAV

Josip Kasac, Denis Kotarski, and Petar Piljek

ABSTRACT

In this paper, a frequency-shifting-based (FSB) algebraic approach to stable on-line parameter identification and state estimation is proposed. The proposed simultaneous parameter identification and state estimation algebraic approach are applied to multirotor adaptive-like tracking control assuming that only position measurement is available. The proposed algebraic approach provides very fast convergence towards true values of system parameters and states, without transients that depend on initial conditions and without peaking phenomenon which is characteristics of high-gain observers. The efficiency of the proposed algorithm is illustrated by a simulation example.

Key Words: Algebraic parameter identification, algebraic state estimation, UAV, multirotor control

I. INTRODUCTION

The implementation of the tracking control laws to Unmanned Aerial Vehicles (UAVs) like multirotors requires the position and velocities measurements. The position measurement is often provided by the motion capture system [1], while the velocity can be measured by an optical flow sensor, for example. However, the velocity sensors are usually avoided in UAVs applications, because of increasing costs or because of a high level of noise in the measured signal. Also, the possibility of failure is increased when more sensors are used. In practice, the velocity signal is usually produced by integrating the accelerometer signal or by differentiating the position measurement.

The main drawback of the accelerometer usage is an inevitable presence of drift due to large errors in the velocity estimate when integrating the unknown drift component [2]. On the other hand, differentiation of the position signal can significantly amplify the measurement noise. If both sensors are available, their fusion may be beneficial [3]. The direct differentiation can be avoided by using state observers [4] but knowledge or identification of system parameters is necessary in that case. The on-line parameter identification is also necessary for UAV tasks like transportation of loads with unknown mass and inertia moments.

A new approach to parameter identification based on algebraic derivative method has been proposed relatively recently [5, 6] for fast and reliable parameter estimation in feedback control systems. The algebraic identification method provides parameters determination in the form of an exact static formula which is based only on measurable input and output variables. The parameter calculation expressions are obtained via algebraic manipulations based on the derivative operator in the operational domain. Unlike traditional methods, the obtained estimator is non-asymptotic: the convergence towards true values of the system parameters is almost instantaneous. Furthermore, the algebraic estimators do not need statistical knowledge of the measurement noise, neither it requires the classical persistency of excitation condition.

Several successful applications and experimental verifications of algebraic derivative approach have

Josip Kasac and Petar Piljek are with the University of Zagreb, Faculty of Mechanical Engineering and Naval Architecture, I. Lucica 5, 10000 Zagreb, Croatia (e-mail: josip.kasac@fsb.hr, petar.piljek@fsb.hr).
Denis Kotarski is with the Karlovac University of Applied Sciences, Trg J.J.Strossmayera 9, Karlovac, Croatia (e-mail: denis.kotarski@vuka.hr).
Josip Kasac is corresponding author.

been reported in the literature. In [7], the algebraic estimation algorithm is applied for the identification of the parameters of a permanent magnet stepper motor and a magnetic bearing. In [8], an on-line algebraic identification methodology for parameter and signal estimation in vibrating mechanical system is experimentally verified.

The algebraic derivative approach is also applied to the state estimation [9, 10, 11, 12]. In [13], the algebraic derivative method is applied for the derivative estimation of noisy signals. In [14], a comparison between an algebraic parameter identification algorithm and classical asymptotic observers for a load of a boost converter is presented. For more applications of the parameter and state estimation in feedback control systems, the interested reader is referred to [6].

Despite the all mentioned advantages of algebraic parameter estimation method, a serious drawback still persists, especially for applications in closed-loop on-line identification. Application of algebraic derivatives for the elimination of initial conditions in combination with invariant filtering in the form of a chain of integrators lead to an unstable time-varying state-space realization of estimator filters. Since the estimator variables are unbounded, an additional switch-off mechanism is necessary after a short period of time.

The problem of inherent instability of algebraic derivative-based estimators is resolved in [15], where a frequency-shifting-based (FSB) algebraic approach is proposed, providing stable on-line parameter identification without needs for periodic re-initialization, like in the case of the conventional algebraic estimators. The proposed FSB algebraic approach is especially suitable for applications in closed-loop on-line identification where the stable behavior of the estimators is a necessary requirement.

In this paper, an FSB algebraic approach for simultaneous parameter identification and observer design is proposed, with application to multirotor control. In comparison with the previous results, the main contribution of this article is the stable algebraic observer design which provides denoising of the measured position signal and estimation of multirotor velocity. Second contributions is a modified version of the FSB algebraic parameter identification method, which provides a reduction of the number of estimator tuning parameters, in comparison with the original approach [15]. The key feature of the proposed estimators, compared with the algebraic derivative-based estimators [6], is the stable state-space realization of the estimator filters without needs for periodic re-initialization.

This paper is organized as follows. In Section II, the dynamic model of the multirotor is presented and certainty-equivalence control problem is formulated. The algebraic parameter identification algorithm based on difference operators in Laplace domain is presented in Section III. The algebraic state observer is presented in Section IV. The simulation results are presented in Section V. Finally, the concluding remarks are emphasized in Section VI.

II. MULTIROTOR DYNAMIC MODEL

The dynamic equations describing the altitude and the attitude motions of a multirotor are basically same as those describing a rotating rigid body with six degrees of freedom [16].

2.1. Kinematics and dynamics of rigid body

The rigid body rotational kinematics equations are given by

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\Omega}_B \boldsymbol{\omega}, \tag{1}$$

where $\boldsymbol{\eta} = [\phi\ \theta\ \psi]^T$ are Euler angles defined according to the xyz-convention, $\boldsymbol{\omega} = [p\ q\ r]^T$ is the angular velocity vector, and

$$\boldsymbol{\Omega}_B = \frac{1}{c_\theta} \begin{bmatrix} c_\theta & s_\phi s_\theta & c_\phi s_\theta \\ 0 & c_\phi c_\theta & -s_\phi c_\theta \\ 0 & s_\phi & c_\phi \end{bmatrix},$$

is the transformation matrix from body to inertial coordinate frame, where $c_{\eta_i} \equiv \cos(\eta_i)$ and $s_{\eta_i} \equiv \sin(\eta_i)$ for $i = 1, 2, 3$, where η_i are elements of the vector $\boldsymbol{\eta} = [\phi\ \theta\ \psi]^T$.

The rigid body translational kinematics equations are given by

$$\dot{\boldsymbol{x}} = \mathbf{R}(\boldsymbol{\eta})\boldsymbol{v}, \tag{2}$$

where $\boldsymbol{x} = [x\ y\ z]^T$ is the vector of translational positions in inertial coordinate frame, $\boldsymbol{v} = [u\ v\ w]^T$ is the linear velocity, and $\mathbf{R} \equiv \mathbf{R}(\boldsymbol{\eta})$ is the rotation matrix from the body frame into the inertial frame, given by

$$\mathbf{R} = \begin{bmatrix} c_\theta c_\psi & c_\psi s_\phi s_\theta - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\theta s_\psi & c_\phi c_\psi + s_\phi s_\psi s_\theta & c_\phi s_\psi s_\theta - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}.$$

The rigid body rotational kinematics can be also represented by the use of rotational matrix,

$$\dot{\mathbf{R}}(\boldsymbol{\eta}) = \mathbf{R}(\boldsymbol{\eta})\mathbf{S}(\boldsymbol{\omega}), \tag{3}$$

where

$$\mathbf{S}(\boldsymbol{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \tag{4}$$

for some $\boldsymbol{a} = [a_1\ a_2\ a_3]^T$.

The three-axis rotational dynamic of rigid body in body-fixed reference frame is given by

$$\mathbf{I}_B\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_B\boldsymbol{\omega}) = \boldsymbol{\tau}, \tag{5}$$

where $\mathbf{I}_B = \mathrm{diag}\{I_x, I_y, I_z\}$ is the diagonal inertia matrix, m is the mass of multirotor and $\boldsymbol{\tau}$ is the vector of actuator torques.

The translational dynamic model of rigid body in the body-fixed reference frame is given by

$$m[\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v}] = \boldsymbol{F} - mg\mathbf{R}(\boldsymbol{\eta})^T\boldsymbol{e}_3, \tag{6}$$

where $\boldsymbol{F}$ is the vector of the actuator forces, g is the gravity acceleration in the inertial frame and $\boldsymbol{e}_3 = [0\ 0\ 1]^T$. The second term on the right-hand side is projections of the gravity force in the body-fixed reference frame.

By using the matrix representation of the vector product $\boldsymbol{a} \times \boldsymbol{b} = \mathbf{S}(\boldsymbol{a})\boldsymbol{b}$, where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ and $\mathbf{S}(\boldsymbol{a})$ defined by Eq. (4), the rigid body dynamic can be rewritten as

$$m[\dot{\boldsymbol{v}} + \mathbf{S}(\boldsymbol{\omega})\boldsymbol{v}] = \boldsymbol{F} - mg\mathbf{R}(\boldsymbol{\eta})^T\boldsymbol{e}_3, \tag{7}$$

$$\mathbf{I}_B\dot{\boldsymbol{\omega}} + \mathbf{S}(\boldsymbol{\omega})\mathbf{I}_B\boldsymbol{\omega} = \boldsymbol{\tau}. \tag{8}$$

Applying the time-derivative of the expression (2) and by using Eq. (3) and (7), the second-order translational dynamic can be obtained

$$\ddot{\boldsymbol{x}} = \frac{1}{m}\mathbf{R}(\boldsymbol{\eta})\boldsymbol{F} - g\boldsymbol{e}_3. \tag{9}$$

During the multirotor navigation with a moderate velocity, the roll and pitch angles remain near zero degrees to allow the approximation of matrix $\boldsymbol{\Omega}_B$ with the identity matrix, thus the derivation of the Euler angles vector $\dot{\boldsymbol{\eta}}$ can be approximated by the body axis angular velocity $\boldsymbol{\omega}$. Under these assumptions, the rotational dynamic model of the multirotor can be reduced to

$$\ddot{\boldsymbol{\eta}} = \mathbf{I}_B^{-1}\boldsymbol{\tau}. \tag{10}$$

The second-order dynamic models (9) and (10) are more appropriate for the control system design and for the algebraic estimator and observer design.

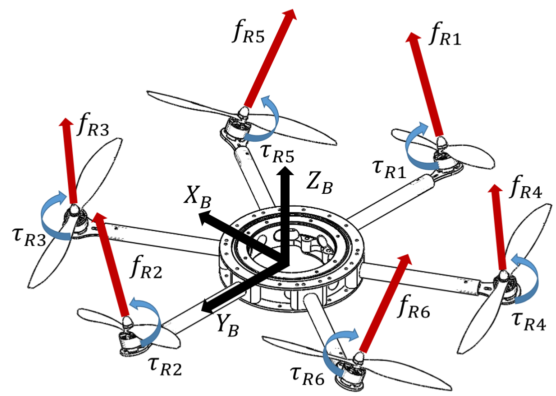

Fig. 1. Fully actuated hexarotor configuration.

2.2. Control of passively tilted multirotor

In [17, 18], a design and control of fully actuated passively tilted multirotor are proposed. The non-flat design with passively tilted rotors can overcome the inherent underactuated property of the flat multirotor configurations, as shown in Fig. 1. The passively tilted multirotor is able to achieve full controllability and decoupling position from orientation. This fact has a significant influence on the multirotor controller design. The non-flat configuration provides six independent control variables, the one for each degree of freedom, contrary to the flat configuration, which provides only four independent control variables [19].

The feedback control law based on the fully actuated multirotor, which provides asymptotic tracking of desired trajectory $\boldsymbol{x}_d(t)$ is

$$\boldsymbol{F} = m\mathbf{R}(\boldsymbol{\eta})^T[\ddot{\boldsymbol{x}}_d + g\boldsymbol{e}_3 - \mathbf{K}_D\dot{\tilde{\boldsymbol{x}}} - \mathbf{K}_P\tilde{\boldsymbol{x}}], \tag{11}$$

so that the closed-loop tracking error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_d$ satisfies

$$\ddot{\tilde{\boldsymbol{x}}} + \mathbf{K}_D\dot{\tilde{\boldsymbol{x}}} + \mathbf{K}_P\tilde{\boldsymbol{x}} = 0, \tag{12}$$

which is asymptotically stable for the positive-definite gain matrices $\mathbf{K}_D, \mathbf{K}_P \in \mathbb{R}^{3\times3}$ [20].

Similarly, the control torque which provides asymptotic tracking of desired angles $\boldsymbol{\eta}_d(t)$ is

$$\boldsymbol{\tau} = \mathbf{I}_B[\ddot{\boldsymbol{\eta}}_d - \mathbf{K}_D\dot{\tilde{\boldsymbol{\eta}}} - \mathbf{K}_P\tilde{\boldsymbol{\eta}}], \tag{13}$$

where $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \boldsymbol{\eta}_d$ leads to the same error dynamics as (12).

The implementation of control laws (11) and (13) require the knowledge of the system parameters m, I_x, I_y, I_z and velocities $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{\eta}}$. Since the multirotor positions $\boldsymbol{x}$ and $\boldsymbol{\eta}$ are only available, the system parameters and velocities should be estimated on-line based only on positions measurement.

III. FSB ALGEBRAIC PARAMETER IDENTIFICATION

The second-order dynamic equations (9) and (10) can be rewritten as

$$\ddot{\boldsymbol{y}} = \boldsymbol{k} \circ \boldsymbol{f}(t) + \boldsymbol{h}, \tag{14}$$

where $\boldsymbol{y} = [\boldsymbol{x}^T \ \boldsymbol{\eta}^T]^T$ is the vector of measurable positions, $\boldsymbol{f}(t) = [(\mathbf{R}(\boldsymbol{\eta})\boldsymbol{F})^T \ \boldsymbol{\tau}^T]^T$ is the nonlinear vector function of the measurable state $\boldsymbol{\eta}$ and known external forces and torques, $\boldsymbol{h} = [-g\boldsymbol{e}_3 \ \mathbf{0}]^T$, is the constant known drift vector, $\boldsymbol{k} = [m^{-1} \ m^{-1} \ m^{-1} \ I_x^{-1} \ I_y^{-1} \ I_z^{-1}]^T$ is the unknown vector of multirotor parameters and $\circ$ denotes the Hadamard pointwise product of vectors, which is defined as $\boldsymbol{x} \circ \boldsymbol{y} = [x_1 y_1 \ \ x_2 y_2 \ \cdots \ x_n y_n]^T$ for some $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. The above expression can be rewritten by components

$$\ddot{y}_i = k_i f_i(t) - g\delta_{i3}, \quad i = 1, 2, ..., 6, \tag{15}$$

where δ_{ij} is Kronecker delta. For the determination of the four parameters m, I_x, I_y, I_z, it is enough to select four of six equation in (15): $\ddot{y}_i = k_i f_i(t)$, for $i = 1, 4, 5, 6$. Since the determination of the parameter k_i depends only on the i-th equation, and to avoid cumbersome notation, the index i will be dropped from the previous expressions in the rest of the paper.

The Laplace transform of the differential equation $\ddot{y} = kf(t)$ is $s^2 y(s) - sy_0 - v_0 = kf(s)$, or

$$s^2 y(s) - kf(s) = R(s), \tag{16}$$

where $y(s) = \mathcal{L}\{y(t)\}$, $f(s) = \mathcal{L}\{f(t)\}$, $R(s) = y_0 s + v_0$, y_0 is the initial position and v_0 is the unknown initial velocity.

3.1. Finite difference operator in Laplace domain

The first step of the algebraic method is the annihilation of the polynomial function $R(s)$ in Eq. (16), which contains initial conditions. The annihilation of initial conditions by using conventional algebraic derivatives in the Laplace domain leads to the unstable estimator filter realization [6]. Instead of algebraic derivatives-based (ADB) method, we will use the finite difference operator in the Laplace domain [15], which provides stable estimator filter realization. In the rest of the paper, we use a notation where functions with the complex variable s represent the Laplace transforms of the functions in the time domain, e.g. $f(s) = \mathcal{L}\{f(t)\}$.

The finite difference operator in Laplace domain of a function $f(s)$ is defined as follows

$$\delta_q f(s) = f(s+q) - f(s). \tag{17}$$

The operator δ_q can be represented as

$$\delta_q = e^{q\frac{d}{ds}} - 1, \tag{18}$$

where $e^{q\frac{d}{ds}}$ is the shift operator with property $e^{q\frac{d}{ds}} f(s) = f(s+q)$. The difference operator decreases the order of polynomials for one degree, so that $\delta_q^n s^{n-1} = 0$. In other words, for the annihilation of polynomial function $R(s) = y_0 s + v_0$, the operator δ_q should be applied two times,

$$\delta_q R(s) = R(s+q) - R(s) = y_0 q, \tag{19}$$
$$\delta_q^2 R(s) = R(s+2q) - 2R(s+q) + R(s) = 0. \tag{20}$$

Since the application of difference operator in Laplace domain produces shifted functions like $f(s+q)$, the frequency shifting property of Laplace transform

$$f(s+q) = \mathcal{L}\{e^{-qt} f(t)\}, \tag{21}$$

will be used for the inverse transform of algebraic expressions to the time domain [21].

3.2. FSB algebraic parameter identification

The first step in the identification of unknown parameter k is the elimination of polynomial $R(s)$ with coefficients which depend on unknown initial conditions.

By applying the operator δ_q^2 on (16), the following expression is obtained: $z_2(s) - kz_0(s) = 0$, where

$$z_0(s) = f(s+2q) - 2f(s+q) + f(s), \tag{22}$$
$$z_2(s) = s^2 w_1(s) + s w_2(s) + w_3(s), \tag{23}$$

and

$$w_1(s) = y(s+2q) - 2y(s+q) + y(s), \tag{24}$$
$$w_2(s) = 4q[y(s+2q) - y(s+q)], \tag{25}$$
$$w_3(s) = 2q^2[2y(s+2q) - y(s+q)]. \tag{26}$$

Now, the expression $z_2(s) - kz_0(s) = 0$ can be rewritten as

$$s^2 w_1(s) + s w_2(s) + w_3(s) - kz_0(s) = 0. \tag{27}$$

In order to overcome effects of high-frequency noise in the measurement of the output variable we must avoid the time-derivatives of output variable, which are represented by terms $s^i w_j(s) = \mathcal{L}\{w_j^{(i)}(t)\}$ in Eq. (27).

By multiplying (27) with $G(s)^3$, where

$$G(s) = \frac{1}{s+\lambda}, \tag{28}$$

is the low-pass invariant filter transfer function with the cut-off frequency $\lambda > 0$, we get

$$\hat{z}_2(s) - k\hat{z}_0(s) = 0, \tag{29}$$

where

$$\hat{z}_0(s) = G(s)^3 z_0(s) = G(s)\{G(s)[G(s)z_0(s)]\}, \tag{30}$$
$$\hat{z}_2(s) = G(s)w_1(s) + G(s)^2\bar{w}_2(s) + G(s)^3\bar{w}_3(s), \tag{31}$$

and

$$\bar{w}_2(s) = w_2(s) - 2\lambda w_1(s), \tag{32}$$
$$\bar{w}_3(s) = w_3(s) - \lambda w_2(s) + \lambda^2 w_1(s). \tag{33}$$

Based on expressions (30) and (31), the following state variables in complex domain are introduced

$$x_1(s) = G(s)[w_1(s) + x_2(s)], \tag{34}$$
$$x_2(s) = G(s)[\bar{w}_2(s) + x_3(s)], \tag{35}$$
$$x_3(s) = G(s)\bar{w}_3(s), \qquad x_4(s) = G(s)x_5(s), \tag{36}$$
$$x_5(s) = G(s)x_6(s), \qquad x_6(s) = G(s)z_0(s), \tag{37}$$

and outputs are $\hat{z}_2(s) = x_1(s)$ and $\hat{z}_0(s) = x_4(s)$.

In the time domain the above expressions become a set of stable linear differential equations in the Jordan canonical form

$$\begin{array}{ll} \dot{x}_1 = -\lambda x_1 + x_2 + w_1, & \dot{x}_4 = -\lambda x_4 + x_5, \\ \dot{x}_2 = -\lambda x_2 + x_3 + \bar{w}_2, & \dot{x}_5 = -\lambda x_5 + x_6, \\ \dot{x}_3 = -\lambda x_3 + \bar{w}_3, & \dot{x}_6 = -\lambda x_6 + z_0, \end{array} \tag{38}$$

which depends only on one tuning parameter λ. The output equations are $\hat{z}_2(t) = x_1(t)$, $\hat{z}_0(t) = x_4(t)$, and the input functions are

$$z_0(t) = h(t)^2 f(t), \qquad w_1(t) = h(t)^2 y(t), \tag{39}$$
$$\bar{w}_2(t) = w_2(t) - 2\lambda w_1(t), \tag{40}$$
$$\bar{w}_3(t) = w_3(t) - \lambda w_2(t) + \lambda^2 w_1(t), \tag{41}$$

where $h(t) = 1 - e^{-qt}$, and

$$w_2(t) = 4q[e^{-2qt} - e^{-qt}]y(t), \tag{42}$$
$$w_3(t) = 2q^2[2e^{-2qt} - e^{-qt}]y(t). \tag{43}$$

Although the quotient $k = \hat{z}_2(t)/\hat{z}_0(t)$ is not affected by the invariant filtering, this expression can not be used directly since the denominator $\hat{z}_0(t)$ can cross the singular value of zero. In order to avoid that the denominator $\hat{z}_0(t)$ cross the singular value of zero, an additional invariant nonlinear filtering is proposed in [22]. By taking the integral of the absolute value of the numerator and denominator, the fraction holds invariant

$$k = \left(\int_0^t |\hat{z}_2(\tau)|d\tau\right)\left(\int_0^t |\hat{z}_0(\tau)|d\tau\right)^{-1}, \tag{44}$$

and the denominator of Eq. (44) is always strictly positive for $t \geq \varepsilon > 0$, where ε is some small positive parameter.

The presented algebraic approach reduces the number of the filter tuning parameters, in comparison with the original approach [15], since the different poles of the invariant filter are replaced with only one multiple pole. By applying the presented algebraic method to the n-th order linear system, the overall number of filter tuning parameters is reduced from $n+2$ to only two parameters, q and λ.

Note that the proposed algorithms can estimate only parameters which are constant in time. In the case of an abrupt change of parameters from the one constant value to another, which is characteristics for the actuator faults [23, 24], the presented algebraic identification methods cannot be applied without modifications like periodic reinitialization [6]. After each reinitialization, the identification process starts from the beginning providing detection of the parameters changes.

IV. FSB ALGEBRAIC APPROACH TO OBSERVER DESIGN

The velocity $\dot{x}_i$ of the system (15) can be estimated by using the similar FSB algebraic approach as in the case of the parameter identification. The assumption is that the parameter k is known or identified on-line using the algebraic approach presented in the previous section.

4.1. FSB algebraic observer

The first step in the algebraic observer design is also the annihilation of initial conditions. The annihilation operator (18) with the real frequency shift is used in combination with the similar invariant filtering procedure as in Subsection 3.2.

By multiplying (27) with $G(s)^2$, we get

$$w_1(s) + G(s)\bar{w}_2(s) + G(s)^2\tilde{w}_3(s) = 0, \tag{45}$$

where $\tilde{w}_3(s) = \bar{w}_3(s) - kz_0(s)$. The above expression can be rewritten as

$$w_1(s) = G(s)[-\bar{w}_2(s) - G(s)\tilde{w}_3(s)]. \tag{46}$$

Furthermore, by multiplying (27) with $G(s)$ we get

$$(s+\lambda)w_1(s) + \bar{w}_2(s) + G(s)\tilde{w}_3(s) = 0, \tag{47}$$

or

$$w_1^{(1)}(s) = -\lambda w_1(s) - \bar{w}_2(s) - G(s)\tilde{w}_3(s), \tag{48}$$

where we introduce the notation $w_1^{(1)}(s) = sw_1(s)$. Based on Eq. (46) and (48), the following state variables are introduced

$$x_2(s) = G(s)[-\bar{w}_3(s) + kz_0(s)], \qquad (49)$$
$$x_1(s) = G(s)[-\bar{w}_2(s) + x_2(s)], \qquad (50)$$

so that expressions (46) and (48) become

$$w_1(s) = x_1(s), \qquad (51)$$
$$w_1^{(1)}(s) = -\lambda w_1(s) - \bar{w}_2(s) - x_2(s). \qquad (52)$$

In time domain the above expressions become a set of stable linear differential equations in the Jordan canonical form

$$\dot{x}_1 = -\lambda x_1 + x_2 - \bar{w}_2(t), \qquad (53)$$
$$\dot{x}_2 = -\lambda x_2 - \bar{w}_3(t) + kz_0(t), \qquad (54)$$

and

$$w_1(t) = x_1(t), \qquad (55)$$
$$w_1^{(1)}(t) = -\lambda w_1(t) - \bar{w}_2(t) - x_2(t), \qquad (56)$$

where $w_1(t)$, $\bar{w}_2(t)$ and $\bar{w}_3(t)$ are defined by Eq. (39)-(41). In the last step of the observer design, we should obtain the explicit expressions for the position and velocity estimate from Eq. (55) and (56) based on the definitions (39)-(41).

From the Eq. (55) and (39), it follows that

$$\hat{y}(t) = \frac{1}{h(t)^2} x_1(t), \qquad (57)$$

where $h(t) = 1 - e^{-qt}$, and $\hat{y}(t)$ is notation for the estimation of the position $y(t)$. The singularity of the above expression in the time instant $t = 0$ can be avoided by the evaluation in $t \geq \varepsilon > 0$, where ε is some small positive parameter. Note that filter (53), (54) and (57) provides denoising of the measured signal $y(t)$.

Further, from Eq. (39) and by using property $\mathcal{L}^{-1}\{w_1^{(1)}(s)\} = \mathcal{L}^{-1}\{sw_1(s)\} = \frac{dw_1(t)}{dt} = w_1^{(1)}(t)$, it follows

$$w_1^{(1)}(t) = 2qe^{-qt}h(t)y(t) + h(t)^2 y^{(1)}(t). \qquad (58)$$

By inserting Eq. (58) and (39)-(43) in Eq. (56), and after some algebraic manipulations, the final expression for the velocity estimation is obtained

$$\hat{y}^{(1)}(t) = (\lambda - 2q)\hat{y}(t) + \frac{x_2(t) + 2qh(t)\hat{y}(t)}{h(t)^2}, \qquad (59)$$

which should be evaluated for $t \geq \varepsilon > 0$.

The estimator filter (53)-(54) has the following linear time-varying state-space realization

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}(t)\boldsymbol{u}, \qquad \hat{\boldsymbol{y}} = \mathbf{C}(t)\boldsymbol{x}, \qquad (60)$$

where $\boldsymbol{x} = [x_1 \ \ x_2]^T$, $\boldsymbol{u} = [y(t) \ \ f(t)]^T$, and $\hat{\boldsymbol{y}} = [\hat{y}(t) \ \ \hat{y}^{(1)}(t)]^T$. The matrices $\mathbf{A}$ and $\mathbf{B}(t)$ are

$$\mathbf{A} = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}, \qquad \mathbf{B}(t) = \begin{bmatrix} b_{11}(t) & 0 \\ b_{21}(t) & b_{22}(t) \end{bmatrix},$$

where $b_{22}(t) = kh(t)^2$, and

$$\begin{aligned} b_{11}(t) &= (2\lambda - 4q)e^{-2qt} + (4q - 4\lambda)e^{-qt} + 2\lambda, \\ b_{21}(t) &= (4q\lambda - 4q^2 - \lambda^2)e^{-2qt} \\ &\quad + (2q^2 + 2\lambda^2 - 4q\lambda)e^{-qt} - \lambda^2. \end{aligned}$$

The matrix $\mathbf{C}(t)$ has the same structure as the matrix $\mathbf{B}(t)$, with the elements $c_{11}(t) = c_{22}(t) = h(t)^{-2}$ and $c_{21}(t) = h(t)^{-2}[\lambda + 2qh(t)^{-1}e^{-qt}]$. Note that $\mathbf{C}(t)$ is nonzero matrix for every $t \geq \varepsilon > 0$.

The state space realization is stable since the characteristic equation $\det(s\mathbf{I} - \mathbf{A}) = (s + \lambda)^2$ has double pole at $s = -\lambda$, where $\lambda > 0$. Further, the all elements of the matrix $\mathbf{B}(t)$ are bounded functions of time.

Now, we will show that the estimator filter (53)-(54) is bounded-input-bounded-output (BIBO) stable system [25], which means that for any bounded inputs $y(t)$ and $f(t)$, the outputs $x_1(t)$ and $x_2(t)$ will be also bounded. The upper bound of the variable $x_2(t)$ can be estimated based on analytical solution and the following chain of inequalities

$$\begin{aligned} |x_2(t)| &= \left| \int_0^t e^{-\lambda(t-\tau)} U_2(\tau) d\tau \right| \\ &\leq \int_0^t \left| e^{-\lambda(t-\tau)} \right| |U_2(\tau)| \, d\tau \\ &\leq \sup_{0 \leq \tau \leq t} |U_2(\tau)| \int_0^t \left| e^{-\lambda\tau} \right| d\tau, \end{aligned}$$

where $U_2(t) = b_{21}(t)y(t) + b_{22}(t)f(t)$. Since the integral in above expression is convergent and less than $1/\lambda$ for any $t \geq 0$, it follows

$$\sup_{t \geq 0} |x_2(t)| \leq \gamma_{21} \sup_{t \geq 0} |y(t)| + \gamma_{22} \sup_{t \geq 0} |f(t)|, \qquad (61)$$

where

$$\gamma_{21} = \lambda^{-1} \max_{t \geq 0} |b_{21}(t)| = \lambda^{-1} \max\{\lambda^2, 2q^2\}, \qquad (62)$$
$$\gamma_{22} = \lambda^{-1} \max_{t \geq 0} |b_{22}(t)| = \lambda^{-1} k. \qquad (63)$$

Similar analysis can be applied on the estimation of upper bound of variable $x_1(t)$

$$\sup_{t\geq 0}|x_1(t)| \leq \gamma_{11}\sup_{t\geq 0}|y(t)| + \gamma_{12}\sup_{t\geq 0}|f(t)|, \quad (64)$$

where $\gamma_{11} = \lambda^{-1}(2\lambda + \gamma_{21})$ and $\gamma_{12} = \lambda^{-1}\gamma_{22}$.

Since the all gains γ_{ij}, $i, j = 1, 2$, in (61) and (64) are finite constants, we conclude that the system is BIBO stable.

4.2. Comparison with ADB observer

The conventional approach to the algebraic observer design is based on the derivative operator with respect to complex frequency s for the elimination of the initial conditions [6]. By differentiating the expression (16) two times with respect to the complex variable s, the following expression is obtained

$$s^2\frac{d^2y(s)}{ds^2} + 4s\frac{dy(s)}{ds} + 2y(s) - k\frac{d^2f(s)}{ds^2} = 0. \quad (65)$$

By multiplying the previous expression with s^{-1} and s^{-2}, two algebraic equations in the complex domain are obtained, which can be represented in time domain by the following state-space system

$$\dot{x}_1 = 2y(t) - kt^2 f(t), \qquad \dot{x}_2 = x_1 - 4ty(t), \quad (66)$$

and output equations

$$\hat{y}(t) = -\frac{1}{t^2}x_2, \qquad \hat{y}^{(1)}(t) = \frac{2}{t}\hat{y}(t) - \frac{1}{t^2}x_1. \quad (67)$$

The system (66) and (67) can be represented by the state-space matrix form (60), where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 2 & -kt^2 \\ -4t & 0 \end{bmatrix}, \ \mathbf{C} = -\frac{1}{t^2}\begin{bmatrix} 0 & 1 \\ 1 & \frac{2}{t} \end{bmatrix}.$$

In the case of ADB observer, the state space realization is unstable since the characteristic equation $\det(s\mathbf{I} - \mathbf{A}) = s^2$ has double pole at $s = 0$. Further, the elements $b_{12}(t) = -kt^2$ and $b_{21}(t) = -4t$ of the matrix $\mathbf{B}(t)$ are unbounded function of time, and matrix $\mathbf{C}(t)$ vanishes as $t \to \infty$.

Applying a similar procedure as in the case of FSB observer, the following upper bound estimates of the state variables are obtained

$$|x_1(t)| \leq 2t\sup_{0\leq\tau\leq t}|y(\tau)| + \frac{k}{3}t^3\sup_{0\leq\tau\leq t}|f(\tau)|, \quad (68)$$

$$|x_2(t)| \leq 3t^2\sup_{0\leq\tau\leq t}|y(\tau)| + \frac{k}{12}t^4\sup_{0\leq\tau\leq t}|f(\tau)|. \quad (69)$$

We can see that the upper bounds increase polynomially in time, which means that the ADB filter realization is not BIBO stable. This is the reason why the periodic resetting is necessary for the implementation of the ADB observer, while it is not necessary in the case of the FSB observer.

V. SIMULATION RESULTS

The algebraic estimators presented in the previous sections will be illustrated on the problem of the multirotor trajectory tracking control in the horizontal plane. We suppose that the torque controller provides attitude stabilization around zero angular positions and that altitude controller provides hovering of multirotor. Further, we assume that the mass of multirotor is unknown and that only noisy position measurement is available.

The control task is the tracking of the time-varying spiral-like reference trajectory

$$\begin{aligned} x_d(t) &= \left(9 - 8e^{-0.2t}\right)\sin((2 - e^{-t})t), \\ y_d(t) &= \left(9 - 8e^{-0.2t}\right)\cos((2 - e^{-t})t). \end{aligned}$$

In that case, control forces in horizontal plane are

$$F_x = \frac{1}{\hat{k}}[\ddot{x}_d - K_D(\hat{x}^{(1)} - \dot{x}_d) - K_P(\hat{x} - x_d)], \quad (70)$$

$$F_y = \frac{1}{\hat{k}}[\ddot{y}_d - K_D(\hat{y}^{(1)} - \dot{y}_d) - K_P(\hat{y} - y_d)], \quad (71)$$

where $\hat{k} = 1/\hat{m}$ is the estimated parameter, $\hat{x}$, $\hat{y}$ are estimated positions and $\hat{x}^{(1)}$, $\hat{y}^{(1)}$ are estimated velocities. The controller gains are $K_D = 5$ and $K_P = 6$, the unknown parameter which should be identified is $k = 2\text{kg}^{-1}$ and the nominal value of the parameter, which is used in controller during the time interval $0 \leq t < \varepsilon = 1\text{s}$, is $k_0 = 1\text{kg}^{-1}$. The tuning parameters of the parameter estimator and observer are $\lambda = 2$ and $q = \lambda/2$.

Fig. 2 shows the measured position signals $x_m(t) = x(t) + 0.5\xi(t)$ and $y_m(t) = y(t) + 0.5\xi(t)$, where $\xi(t)$ is some Gaussian noise of standard normal distribution $\mathcal{N}(0, 1)$, which are inputs of parameter and state estimators presented in the previous sections.

Fig. 3 shows parameter k obtained by the FSB estimator in the case with and without measurement noise. We can see that in time instant $\varepsilon = 1$ s when the estimators are switched on, the convergence towards real values of the parameter k is almost instantaneous.

Fig. 4 and Fig. 5 show estimated positions and velocities of the multirotor in the case with

Table 1. Estimated values and relative errors of parameter and states at the time instant $t_f = 10$s in the case with noise.

	Real value	Estimated value	Relative error (%)
$k(t_f)$	2	1.9994	0.0319
$x(t_f)$	7.2197	7.2312	0.1592
$y(t_f)$	3.2375	3.2241	0.4133
$\dot{x}(t_f)$	6.6683	6.6710	0.0410
$\dot{y}(t_f)$	−14.3529	−14.3569	0.0277

measurement noise. Similarly, as in the case of parameter estimation, the convergence towards real values of the multirotor positions and velocities is almost instantaneous after switching time $\varepsilon = 0.1$s. The estimated position and velocity signals are well denoised thanks to low-pass invariant filtering with small cut-off frequency λ. Table 1 provides numerical values of the estimated parameter, positions, velocities and their relative errors at the time instant of 10 seconds.

Fig. 6 shows the comparison of system response and reference trajectories of the multirotor in the case with measurement noise, while Fig. 7 shows the multirotor and reference trajectory in xy plane. The system response is almost the same as the response of the system with the controller with known parameter and measured velocities.

Fig. 8 and Fig. 9 show comparison results between the proposed FSB observer and conventional algebraic derivative-based (ADB) observer [6]. Estimation errors in the case with noise, as shown in Fig. 8, are very similar. However, the state variables of the ADB observer, shown in Fig. 9, are unbounded in time since the filter state-space realization (66) is not BIBO stable, as follows from Eq. (68) and (69). On the other hand, the state variables of the FSB observer are bounded in time since the filter state-space realization (53)-(54) is BIBO stable, as follows from Eq. (61) and (64).

Note that the convergence properties of the presented algebraic estimators do not depend on the choice of the reference trajectory.

VI. CONCLUSIONS

In this paper, an algebraic approach to on-line parameter identification and state estimation is proposed, with application to multirotor control. The proposed approach is based on the difference operators in the Laplace domain. The main benefit of this approach is the stable state-space filter realization

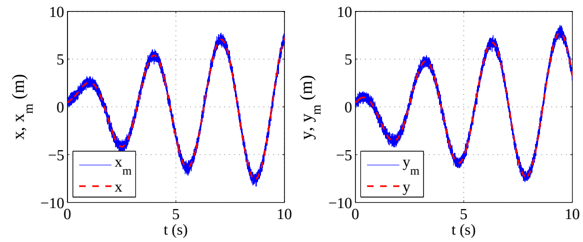

Fig. 2. The positions $x(t)$ and $y(t)$ with measurement noise.

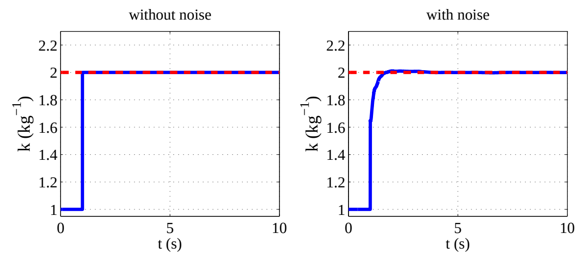

Fig. 3. Estimated parameters in the case with and without measurement noise.

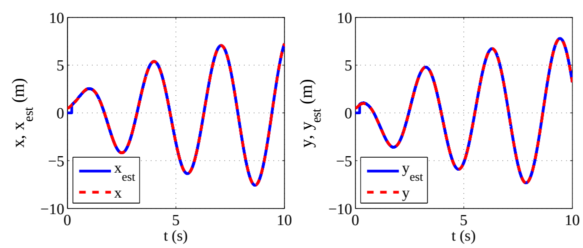

Fig. 4. Estimated positions of the multirotor in the case with measurement noise.

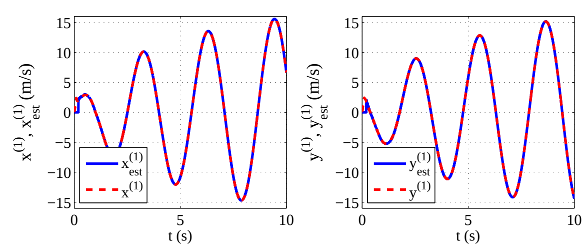

Fig. 5. Estimated velocities of the multirotor in the case with measurement noise.

of the parameter estimators and state observers. The simulation results demonstrate the fast convergence of the estimated parameter and the system states towards

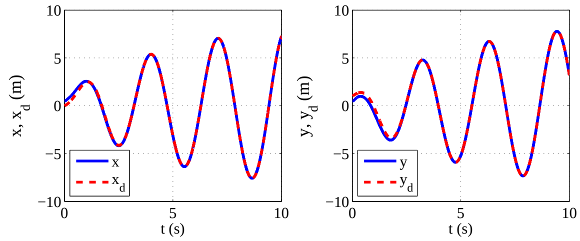

Fig. 6. The system response and reference trajectories of the multirotor in the case with measurement noise.

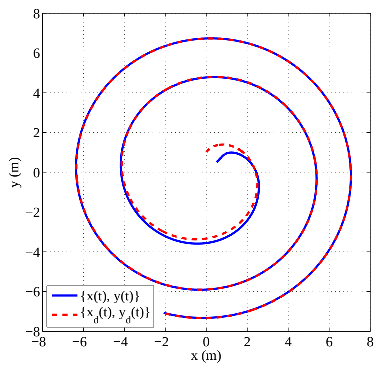

Fig. 7. The trajectory of the multirotor in the case with measurement noise.

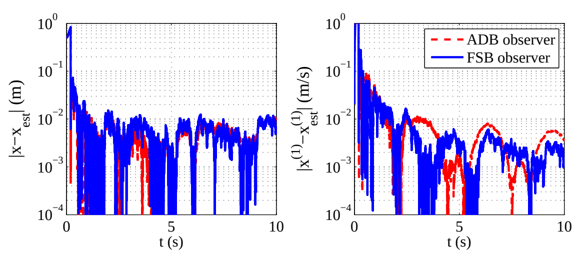

Fig. 8. Estimation errors for ADB and FSB observers in the case with measurement noise.

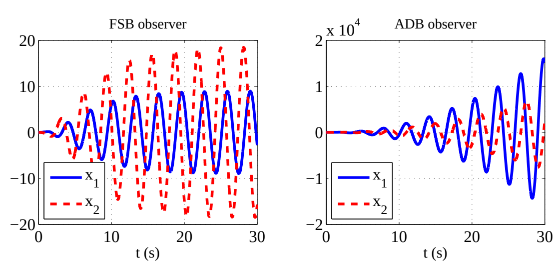

Fig. 9. State variables of ADB and FSB observers in the case with measurement noise.

true values. It is shown that the proposed estimators are insensitive to unknown initial conditions and robust with respect to measurement noise. The future work will be oriented toward the application of the algebraic approach for the estimation of external disturbances and experimental verification on passively tilted hexarotor.

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Josip Kasac is a professor in the Department of Robotics and Automation of Manufacturing Systems at Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Croatia. Dr. Kasac received his Ph.D. in Mechanical Engineering from the University of Zagreb in 2005. His research interests are control of nonlinear mechanical systems, optimal control and algebraic estimation methods.

Denis Kotarski received his BS in Mechanical Engineering from the University of Zagreb in 2011. He is a Ph.D. candidate in the Department of Robotics and Automation of Manufacturing Systems at Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Croatia. His research interests include multirotor UAV dynamics and nonlinear control design.

Petar Piljek received his Ph.D. in Mechanical Engineering from the University of Zagreb in 2017. Since then he has held the post-doctoral research position at the Department of Technology at the Faculty of Mechanical Engineering and Naval Architecture of the University of Zagreb, Croatia. His research interests include metal forming and production technologies, modeling of dynamic systems, and control algorithm implementations.