Ring buckling analysis based on the toroidal shell theory

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RING BUCKLING ANALYSIS BASED ON THE TOROIDAL SHELL THEORY

Summary

The in-plane and the out-of-plane buckling theory of rings exposed to external pressure is presented. A ring is considered as a segment of a toroidal shell. The governing ring equations are obtained by deducing the toroidal shell energy equations for the linear and the nonlinear strain. The obtained formulae for critical load are compared with those known in relevant literature. The critical load depends on the assumption concerning the load behaviour during buckling. Illustrative examples are solved numerically by means of several commercial FEM computer programs in order to investigate which assumptions are introduced in the ring buckling analysis.

Key words: *ring buckling, energy approach, in-plane buckling, out-of-plane buckling, critical load, analytics, FEM*

1. Introduction

Rings are ordinarily used as reinforcing structural elements in shell structures, like submarine pressure hulls, pressure vessels, and cargo tanks in liquefied gas carriers [1]-[8]. The function of rings in these structures is to sustain radial components of loading in the spanning elements, Fig. 1. These radial loads can induce high circumferential stress in the ring, which can cause buckling failure in the case of external load.

A ring is rigidly attached to the shell elements, which provides restraint on translational displacements at the joint. However, this restraint is not present in all structural components. Hence, two types of ring buckling are distinguished. When a ring is radially loaded in its plane, it may buckle by simultaneous in-plane flexure and out-of-plane flexure with torsion. If the ring has an axis of symmetry lying in its plane, the in-plane and the out-of-plane buckling are uncoupled and can be studied independently, [9].

In-plane buckling can be analysed with or without extensional effects. Critical external pressure depends on the assumption related to the pressure direction during buckling, [10]. This problem is analysed in [11]-[19]. An overview of the achieved solutions is given in [20].

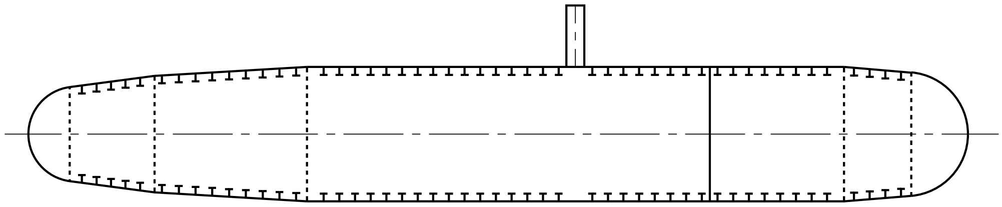


Fig. 1 Submarine pressure hull

The first solution to the problem of the out-of-plane buckling of rings was that achieved by Timoshenko and Gere, [11]. Since then, several different theories of out-of-plane ring buckling have been proposed, [21]-[24]. These theories give quite different results for some problems. A comparison and the testing of these theories are given in [25]. In addition, a new rigorous thin-walled theory for mono-symmetric rings is derived.

Recently, the vibration theory of rotating and pressurized closed toroidal shells has been developed utilizing the Rayleigh-Ritz method and a Fourier series, [26]. The same approach is then used in the development of finite strip method, which makes it possible to analyse the vibration of open toroidal shells with arbitrary boundary conditions, [27]. Moreover, rings for the in-plane and the out-of-plane vibration can be considered as segments of a toroidal shell, [28].

In this paper, the vibration theory of a toroidal shell is adapted to the in-plane and out-of-plane buckling problem of rings. The obtained formulae are compared to the known results of different buckling theories. It is further investigated which assumption is used for the numerical analysis by the available FEM software.

2. Basic expressions for the linear and the nonlinear strain energy of a toroidal shell

The linear and the nonlinear strain energy of a toroidal shell is defined within the vibration theory of rotating and pressurized toroidal shells based on the energy approach, i.e. the Rayleigh-Ritz method, [26]. In the case of buckling analysis, the meridional displacement in the cross-sectional ϑ-plane, u, the circumferential displacement, v, and the radial displacement, w, Fig. 2, can be assumed in the form

$$\begin{aligned} u(\vartheta,\varphi) &= U(\vartheta)\cos n\varphi \\ v(\vartheta,\varphi) &= V(\vartheta)\sin n\varphi \\ w(\vartheta,\varphi) &= W(\vartheta)\cos n\varphi. \end{aligned} \tag{1}$$

Functions $U(\vartheta)$, $V(\vartheta)$ and $W(\vartheta)$ are the amplitudes of displacement which describe the cross-sectional buckling mode profile.

The linear modal strain energy, after integration in the circumferential direction, in the domain $0 \le \varphi \le 2\pi$, reads

$$\begin{aligned} E_s = \int_{\vartheta} \Big[& \frac{1}{2}p_1(U')^2 + \frac{1}{2}p_2U^2 + p_3U'U + \frac{1}{2}p_4(V')^2 + \frac{1}{2}p_5V^2 + p_6V'V \\ & + p_7U'V + p_8UV' + p_9UV \\ & + \frac{1}{2}q_1(W'')^2 + \frac{1}{2}q_2(W')^2 + \frac{1}{2}q_3W^2 + q_4W''W' + q_5W''W + q_6W'W \\ & + q_7W''U' + q_8(W''U + W'U') + q_9W'U + q_{10}WU' + q_{11}WU \\ & + q_{12}W''V + q_{13}W'V' + q_{14}W'V + q_{15}WV' + q_{16}WV \Big] \mathrm{d}\vartheta, \end{aligned} \tag{2}$$

where $p_i(\vartheta), i=1,2...9$ and $q_j(\vartheta), j=1,2...16$ are the variable coefficients which can be found in reference [26].

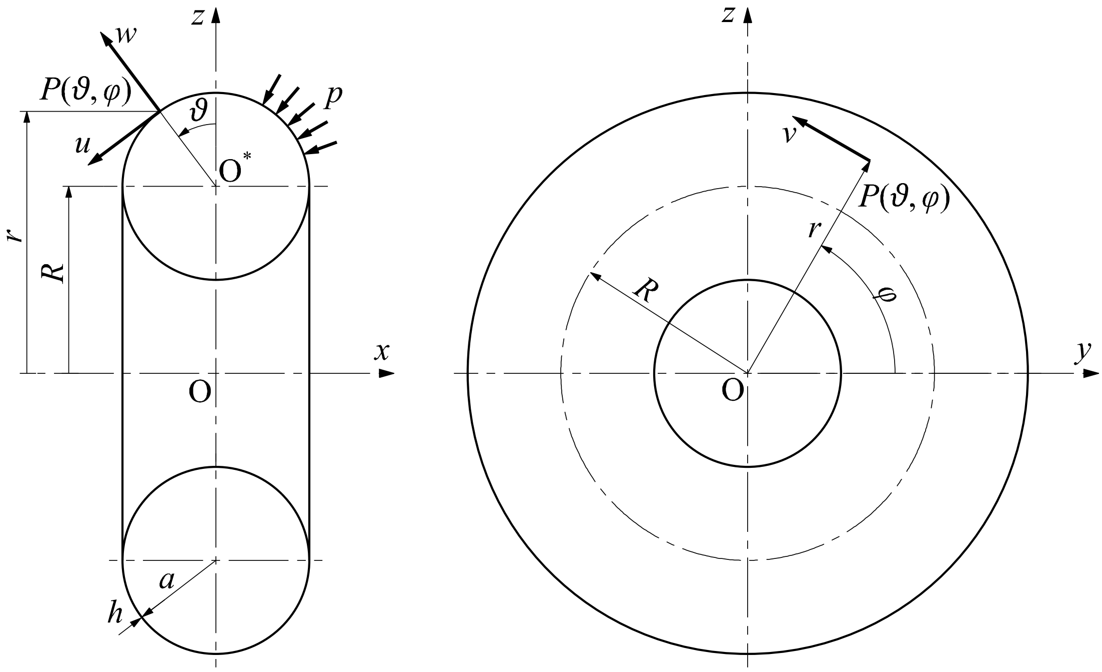


Fig. 2 Closed toroidal shell, main dimensions, load and displacements

The linearised nonlinear strain energy due to pre-stressing is obtained in the form

$$
\begin{aligned}
E_G = \int_{\vartheta} \Big[& \frac{1}{2}c_1\left(U'\right)^2 + \frac{1}{2}c_2U^2 + \frac{1}{2}c_3\left(V'\right)^2 + \frac{1}{2}c_4V^2 + c_5V'V \\
& + \frac{1}{2}c_6\left(W'\right)^2 + \frac{1}{2}c_7W^2 + c_8UV + c_9\left(U'W - UW'\right) + c_{10}UW + c_{11}VW \Big] \mathrm{d}\vartheta,
\end{aligned}
\tag{3}
$$

where $c_i(\vartheta), i=1,2...11$ are the variable coefficients which can be found in [26].

It is necessary to mention that two formulations of the nonlinear strain are known. One is the Green-Lagrange strain and the other is the engineering strain. The latter is the reduction of the former relating to the extensional terms, [29],[30].

3. Ring in-plane buckling

For the purpose of investigating the ring in-plane buckling, a toroidal shell segment in the vicinity of the $\vartheta = \pi/2$ angle is considered, as shown in Fig. 3. Relevant displacements in the in-plane buckling are the circumferential and the radial buckling, V and W. After integration, the expressions for the strain energy and the geometric strain energy, Eqs. (2) and (3) respectively, are no longer functions of the ϑ angle. Therefore, they are reduced to the following form for a unit length in the arch ($b = 1$)

$$
\begin{aligned}
E_S &= \frac{1}{2}p_5V^2 + \frac{1}{2}q_3W^2 + q_{16}VW \\
E_G &= \frac{1}{2}c_4V^2 + \frac{1}{2}c_7W^2 + c_{11}VW.
\end{aligned}
\tag{4}
$$

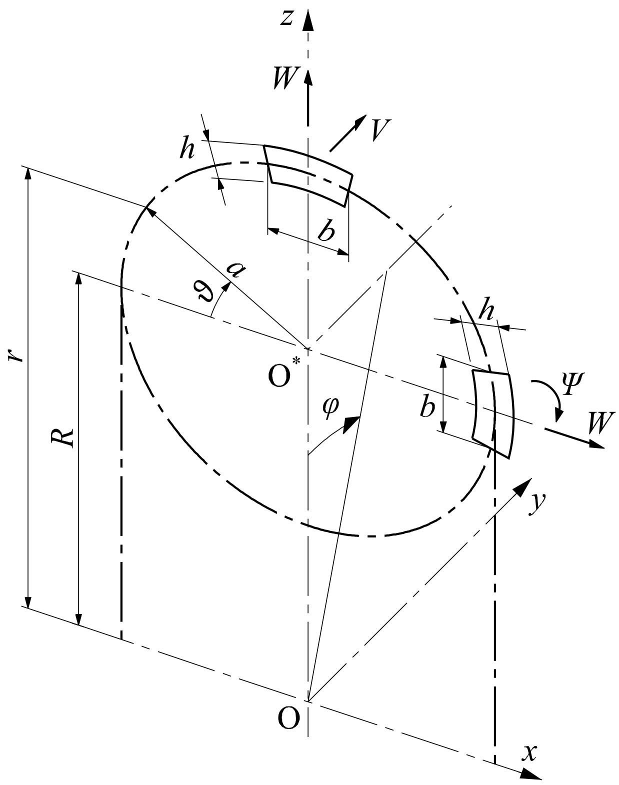


Fig. 3 Rings as segments of a toroidal shell

Coefficients p_i, q_i and c_i in Eq. (4) are specified according to [26], taking into account that $\vartheta = \pi/2$ and $\nu = 0$ for the ring as a one-dimensional structural element

$$\begin{aligned} p_5 &= n^2\left(1+\beta\right) \\ q_3 &= 1+n^4\beta \\ q_{16} &= n\left(1+n^2\beta\right) \\ c_4 &= c_7 = \left(1+n^2\right)\lambda \\ \mathrm{c}_{11} &= 2n\lambda. \end{aligned} \tag{5}$$

Parameters in (5) are

$$\beta = \frac{D}{Kr^2}, \quad \lambda = \frac{\tilde{N}_\varphi}{K}, \tag{6}$$

where $D = EI$ is the bending stiffness, $K = EA$ is the tensional stiffness and $\tilde{N}_\varphi = qr$ is the tensional force due to the pressure load q per unit length. The formulae for coefficients c_4, c_7 and c_{11} shown in (5) (taken from [28]) are related to the Green-Lagrange strains, [29].

A ring loses stability at some critical pressure load. In that moment, the ring suddenly changes its geometry, and the ordinary strain energy, E_S, is transformed into the strain energy due to pre-stressing, E_G, expressed in Eq. (4). Hence, the balance of energy reads

$$\Pi = E_S - E_G, \tag{7}$$

where Π is zero in the case of the exact solution, or it has to be minimum in the case of an approximate solution. By satisfying this condition, one obtains

$$\begin{Bmatrix} \dfrac{\partial \Pi}{\partial V} \\ \dfrac{\partial \Pi}{\partial W} \end{Bmatrix} = \left(\begin{bmatrix} p_5 & q_{16} \\ q_{16} & q_3 \end{bmatrix} - \begin{bmatrix} c_4 & c_{11} \\ c_{11} & c_7 \end{bmatrix} \right) \begin{Bmatrix} V \\ W \end{Bmatrix} = \{0\}. \tag{8}$$

The first matrix in Eq. (8) is the ordinary stiffness matrix, and the second one is the geometric stiffness matrix.

The determinant of Eq. (8) has to be zero, i.e.

$$\text{Det} = (p_5 - c_4)(q_3 - c_7) - (q_{16} - c_{11})^2 = 0. \tag{9}$$

Inserting the equations from (5) into (9), after some manipulation one obtains the following quadratic equation

$$a\lambda^2 + b\lambda + c = 0, \tag{10}$$

where

$$\begin{aligned} a &= \left(1-n^2\right)^2 \\ b &= -\left(1-n^2\right)^2 \left(1+n^2\beta\right) \\ c &= \left(1-n^2\right)^2 n^2\beta. \end{aligned} \tag{11}$$

The solution of the eigenvalue problem from Eq. (10) reads

$$\lambda_{1,2} = \frac{1}{2}\left[\left(1+n^2\beta\right) \pm \left(1-n^2\beta\right)\right], \tag{12}$$

and one finds that

$$\lambda_1 = 1, \quad \lambda_2 = n^2\beta. \tag{13}$$

Now, the determinant defined in (9) can be presented in the form

$$\text{Det} = \left(1-n^2\right)^2 (\lambda - 1)\left(\lambda - n^2\beta\right) = 0. \tag{14}$$

If the mode number $n = 0$, neither deformation nor buckling occurs. The value $n = 1$ is related to the rigid body motion, in which case $V = W$, and there is no buckling. For the elastic modes $n \geq 2$, using (6), one finds from (13) two eigenloads

$$q^{(1)} = \frac{K}{r} = \frac{EA}{r}, \tag{15}$$

$$q^{(2)} = n^2\beta\frac{K}{r} = n^2\frac{EI}{r^3}. \tag{16}$$

The first load, which is very high, is related to extensional buckling. The second load is relevant for bending buckling. The minimal value for $q^{(2)}$ is obtained for $n = 2$, and the critical load reads

$$q_{\text{cr}} = 4\frac{EI}{r^3}. \tag{17}$$

If the engineering strain is used [30], the coefficients of the nonlinear strain, Eq. (5), are reduced to, [28],

$$c_4 = \lambda, \quad c_7 = n^2\lambda, \quad c_{11} = n\lambda. \tag{18}$$

The determinant of the corresponding eigenvalue problem (8) is obtained in the form

$$\mathrm{Det} = \left(1-n^2\right)^2 \left(\lambda - n^2\beta\right) = 0. \tag{19}$$

Comparing Eqs. (14) and (19), it is obvious that the factor λ – 1, related to the extensional buckling, vanishes from (19), and the critical load is the same as in the case of the Green-Lagrange strain, (17).

The buckling parameter $\mu = q_{cr}r^3/(EI)$ determined in this paper is compared with the values known in the relevant literature, Table 1. The value of μ depends on the assumptions made in the buckling analysis:

1. $\mu = 3$: Load remains normal to the deformed ring during buckling.
2. $\mu = 3.265$: Constant-directional load.
3. $\mu = 4$: Direction of load does not change during buckling.
4. $\mu = 4.5$: Load remains directed toward the centre of the ring during buckling.
5. $\mu = 5.6$: Constant-directional load.

In the case of a thick ring, the formula for the buckling load is extended to the following form, [31]

$$q_{\mathrm{cr}} = \mu \frac{EI}{r^3} \frac{1}{1 + 4\dfrac{EI}{GA_S r^2}}, \tag{20}$$

where $G = E/[2(1+\nu)]$ is the shear modulus and $A_S = k_S A$ is the shear area determined as a part of the ring cross-section area, A. The shear coefficient k_S depends on the cross-section shape, [32],[33].

More sophisticated solutions to the problem of thick ring buckling, based on the theory of elasticity, are presented in [34]-[36].

Table 1 Ring in-plane buckling parameter $\mu = q_{cr}r^3/(EI)$

	Present	Literature				
		(1)	(2)	(3)	(4)	(5)
		[11],[12]	[12],[18]	[14]-[17],[22]	[12],[13],[22]	[19]
Formula	n^2	n^2-1		n^2		
$n = 2$	4	3	3.265	4	4.5	5.6

4. Ring out-of-plane buckling

This type of buckling is analysed by considering the toroidal shell segment in the vicinity of the $\vartheta = \pi$ angle with two degrees of freedom, i.e. the deflection W and the twist angle Ψ, Fig. 3. Since extensional displacements U and V are zero, the strain energy and the geometric strain energy according to Eqs. (2) and (3) are reduced to

$$\begin{aligned} E_S &= \frac{1}{2}q_1\left(W''\right)^2 + \frac{1}{2}q_2\left(W'\right)^2 + \frac{1}{2}q_3W^2 + q_4W''W' + q_5W''W + q_6W'W \\ E_G &= \frac{1}{2}c_6\left(W'\right)^2 + \frac{1}{2}c_7W^2 \end{aligned} \tag{21}$$

Coefficients q_i and c_i in Eq. (21) are specified in [26],[28]. In the considered case $c_6 = 0$ and $c_7 = \pi n^2 N_\varphi a/r$, where $N_\varphi = pr$ is the circumferential membrane force per unit length due to external pressure.

The rotation angle in Eq. (21) is expressed as

$$W' = \frac{a}{r} X, \tag{22}$$

and, after some manipulation, for curvature one obtains, [26],

$$W'' = -\frac{a}{r}\frac{q_4}{q_1} X - \frac{a}{r}\frac{q_5}{q_1} W. \tag{23}$$

That way, the eigenvalue problem (7) for the out-of-plane buckling is reduced to

$$\begin{bmatrix} a_{11} - b_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} W \\ X \end{Bmatrix} = \{0\}, \tag{24}$$

where the elements of the above matrix are given in [28]. Element b_{11} contains the membrane force N_φ. Solving Eq. (24) for b_{11} yields

$$b_{11} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}}. \tag{25}$$

By exchanging the toroidal shell geometric parameters for ring parameters, as shown in [28], and by taking into account the total force $\tilde{N}_\varphi = N_\varphi b = pbr = qr$, where b is the height of the ring cross-section, one obtains the following formula for eigenvalue load

$$q = \frac{EI}{r^3} \frac{\left(n^2 - 1\right)^2}{n^2 + \left(1 + \nu\right)}. \tag{26}$$

The critical buckling load is obtained for the first elastic mode, due to the minimum absorbed strain energy, i.e. for $n = 2$

$$q_{\mathrm{cr}} = \frac{EI}{r^3} \frac{9}{4 + \left(1 + \nu\right)}. \tag{27}$$

Formula (27) is similar to the Timoshenko formula, [11], which can be presented in the same form

$$q_{\mathrm{cr}} = \frac{EI}{r^3} \frac{9}{4 + \dfrac{EI}{GJ}}, \tag{28}$$

where $J = k_\mathrm{t} I_\mathrm{p}$ is the torsional modulus, as a part of the polar moment of inertia of the ring cross-section. The torsional coefficient depends on the cross-section shape, [37],[38],[42]. In the case of circular cross-section, $k_\mathrm{t} = 2$ and formula (28) becomes identical to (27).

In literature, there are some more sophisticated buckling theories of rings with complex cross-sections which are considered, [25].

5. Numerical examples

5.1 Ring in-plane buckling

Since there are different analytical formulae for the critical pressure load of the ring in-plane buckling, Table 1, it is interesting to consider solutions obtained by various FEM commercial computer programs. Buckling of a thin ring with the following geometric and physical properties is analysed: $r = 1$ m, $b = 0.1$ m, $h = 0.01$ m, $E = 2.1\cdot 10^{11}$ N/m^2, $p = 1$ MPa (Fig. 4a). Three software packages are used, i.e. *Abaqus* [39], *Catia* [40] and *SolidWorks* [41]. The ring buckling is analysed as a 1D, a 2D and a 3D spatial problem, taking $\nu = 0$. Accordingly, different beam, shell and solid finite elements are employed. The buckling modes determined by *Abaqus* are shown in Fig. 5. In all calculations, the critical buckling parameter μ for $n = 2$ assumes a value of 3 or 4, Table 2. Obviously, different assumptions about the load direction during buckling are applied, as explained in Section 3. Nevertheless, the buckling shapes remain similar regardless of the assumption related to the pressure direction during buckling.

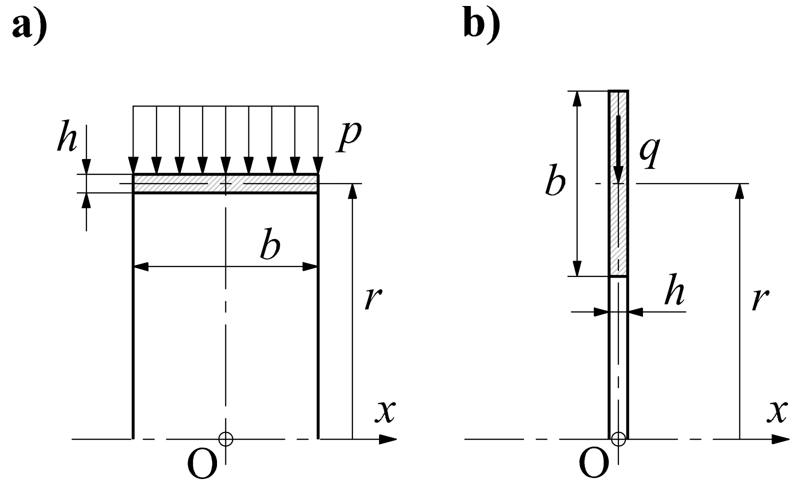


Fig. 4 Rings for: a) in-plane buckling, b) out-of-plane buckling

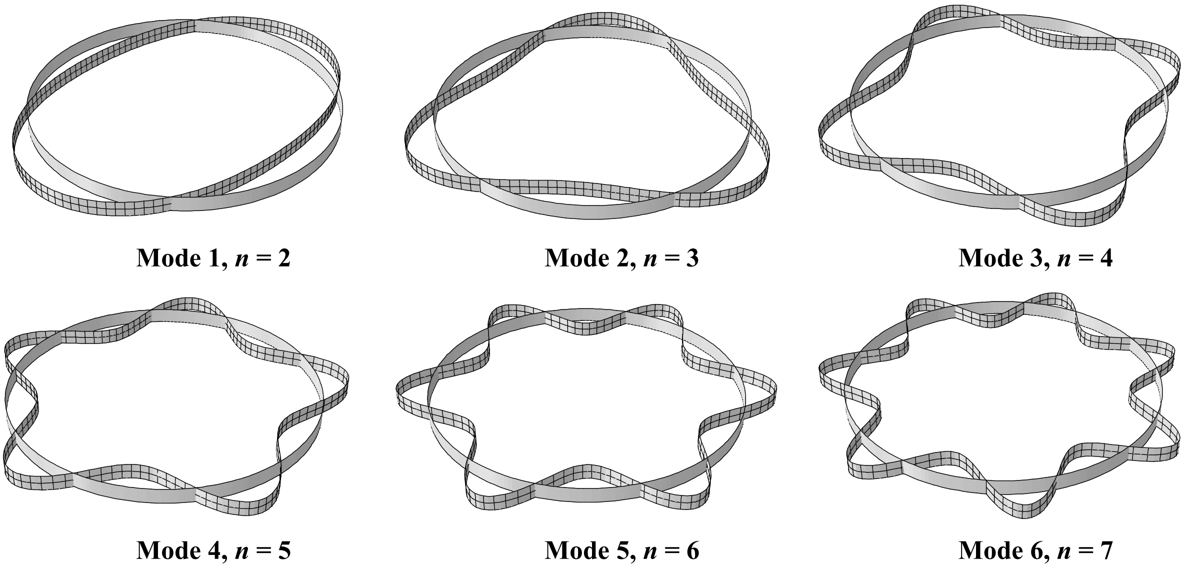


Fig. 5 In-plane buckling modes of a thin ring under external pressure (*Abaqus*)

Table 2 Ring in-plane buckling parameters: $\lambda = p_{cr}/p$, $\mu = p_{cr}br^3/(EI)$, p = 1 MPa

Software	FE type	No. FEs	λ	μ
Abaqus	B32(T)	1D, 125	0.052496	3
	B33(EB)	1D, 125	0.070012	4
	S8R(nd)	2D, 248	0.052495	3
	S8R(nu)	2D, 248	0.069994	4
	C3D20R(nd)	3D, 512	0.052497	3
	C3D20R(nu)	3D, 512	0.069997	4
Catia	BEAM	1D, 126	0.070031	4
	QD8	2D, 252	0.069995	4
	HE20	3D, 800	0.070049	4
SolidWorks	SHELL6	2D, 252	0.070015	4

T – the Timoshenko beam theory

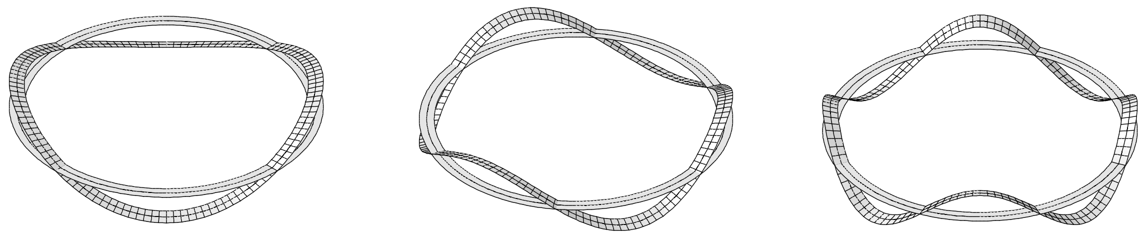
EB – the Euler-Bernoulli beam theory

nd – pressure normal to the deformed ring

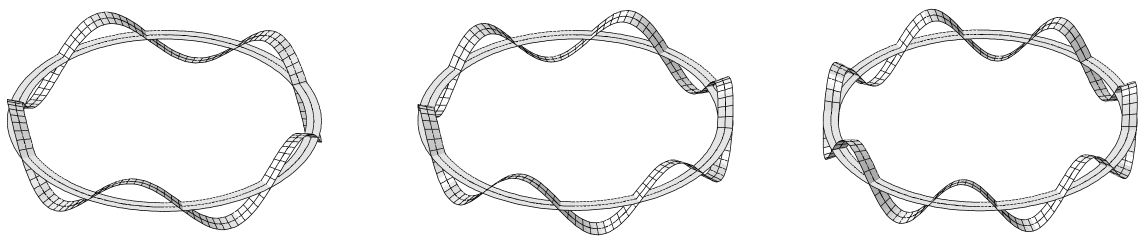
nu – pressure normal to the undeformed ring

5.2 Ring out-of-plane buckling

Buckling of the ring shown in Fig. 4b is analysed. The ring has similar geometric and physical properties as that shown in Fig. 4a: $r = 1$ m, $b = 0.1$ m, $h = 0.01$ m, $E = 2.1\cdot10^{11}$ N/m^2, $q = 10^5$ N/m, $\nu = 0$. Torsional modulus for the rectangular cross-section is determined according to [42], $J = k_1bh^3$, where the coefficient k_1 for the aspect ratio $b/h = 10$ reads 0.312. Numerically determined buckling parameters obtained by *Abaqus*, *Catia* and *SolidWorks* are compared with the analytical value in Table 3. Very good agreement is observed in the cases when pressure remains normal to the undeformed ring ($\mu \approx 2$). However, different results are obtained in the cases when pressure remains normal to the deformed ring ($\mu \approx 3$). Buckling modes determined by *Abaqus* are shown in Fig. 6.



Mode 1, *n* = 2 **Mode 2, *n* = 3** **Mode 3, *n* = 4**



Mode 4, *n* = 5 **Mode 5, *n* = 6** **Mode 6, *n* = 7**

Fig. 6 Out-of-plane buckling modes of a thin ring under external pressure (*Abaqus*)

Table 3 Ring out-of-plane buckling parameters: $\lambda = q_{cr}/q$, $\mu = q_{cr}r^3/(EI)$, $q = 10^5$ N/m, $\nu = 0$, Analytical, Eq. (28): $\lambda = 0.034736$, $\mu = 1.98492$

Software	FE type	No. FEs	λ	μ
Abaqus	B32(T)	1D, 125	0.050864	2.90653
	B33(EB)	1D, 125	0.034791	1.98807
	S8R(nd)	2D, 248	0.050648	2.89417
	S8R(nu)	2D, 256	0.034864	1.99223
	C3D20R(nd)	3D, 512	0.050953	2.91162
	C3D20R(nu)	3D, 512	0.034882	1.99326
Catia	BEAM	1D, 126	0.034760	1.98626
	QD8	2D, 252	0.034861	1.99207
	HE20	3D, 800	0.034930	1.99599
SolidWorks	SHELL6	2D, 536	0.035050	2.00286

T – the Timoshenko beam theory

EB – the Euler-Bernoulli beam theory

nd – pressure normal to the deformed ring

nu – pressure normal to the undeformed ring

6. Conclusion

The toroidal shell theory is universal since it comprises the complete class of shells of revolution, i.e. cylindrical, conical and spherical shells, as well as circular membranes and plates. Moreover, rings for the in-plane and out-of-plane buckling analyses can be considered as segments of a toroidal shell.

By adapting the toroidal shell energy equations for the linear and the nonlinear strain to the in-plane and the out-of-plane ring buckling, a relatively simple eigenvalue problem is formulated. This leads to analytical formulae for critical loads.

In literature, there are different formulae for the ring in-plane buckling; they depend on the assumption relating to the load behaviour during buckling. The derived formulae, with the buckling coefficient $\mu = 4$, correspond to the case of constant pressure direction during buckling.

For the ring out-of-plane buckling, the structure of the formulae for critical load is the same as that given by Timoshenko. However, in that formula, the appropriate torsional modulus of the considered ring cross-section has to be used.

Illustrative examples of ring buckling solved numerically by means of several commercial FEM computer programs show that different assumptions concerning the load behaviour during buckling are introduced.

For the determination of geometrical properties of rings with a very complex thin-walled cross-section, a numerical procedure based on the strip method, used for ship hull cross-sections, can be applied, [43].

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