

# Numerical modeling of coupled dynamical systems

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University of Zagreb

Faculty of Mechanical Engineering and Naval Architecture

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DYNAMICAL SYSTEMS**

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Sveučilište u Zagrebu

Fakultet strojarstva i brodogradnje

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**NUMERIČKO MODELIRANJE  
SPREGNUTIH DINAMIČKIH SUSTAVA**

DOKTORSKI RAD

Mentor:

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# Preface

The last six years or so were dynamic, to say the least. The effort put into research in relation to research progress is highly out of phase, with the largest gain typically occurring in the high-frequency range, i.e. around deadlines. But, in the end, I'm happy to say that I was able to reject the dissipative nature of mundane routine and get back on my feet to learn a plethora of new talents and gain a vast amount of valuable knowledge. Earning a doctorate degree was no easy feat and I hope my dissertation will serve as a valuable resource for others and a moving reminder of how far I've come and for what and to whom I should to be thankful.

First and foremost, I'd like to thank my supervisor, associate professor Marko Jokić, PhD: ever since writing my master's thesis and now throughout the entire PhD course, you haven't once said that you don't have time for me — and all that time you gave me is absolutely indispensable — it helped me with the research, narrowing things down, focusing on the important topics, obtaining meaningful results, and being, above all, a practical engineer and a realist. I would not have made it to the end without your direction and patience — thank you for this wild voyage through my PhD course.

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I'm grateful to my bachelor's degree supervisor and a man I look up to, full professor Zdenko Tonković, PhD. Because of you, I've returned to academia, eager to learn and research, to teach and share knowledge with the same zeal that you do, and to inspire future generations of engineers at FSB. I've learnt from you that being more socially involved, both politically and voluntarily, is important, and that honesty, openness, and everyday kind words to others are vital.

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I'm glad and thankful to have close family and friends. To my mom Ika and dad Niko, who always had high hopes and expectations for me, but you never had any doubts about me. This has made me resilient and inspired to keep going from early days. From you two, I've learned everything I need to know to be a good man and to put knowledge and kindness above everything else. I'm also thankful to my brother Elgar and my sisters Erika, Ema, and Klaudia. From you I have learned to take better care for others. I'm grateful to my mother-in-law Višnja and father-in-law Danimir for accepting me as their own and treating me like family. You taught me to work hard and to see people for who they really are. My sincere appreciation goes to Marija and Igor, my sister and brother-in-law, who welcomed me into the family and were always there to provide a helping hand without expecting anything in return. You have taught me how to remain modest while still pushing myself to my limits.

I'm also thankful to our wedding godfathers Vedran and Tomislav, our wedding godmothers Sara and Ivana, and our children's godparents Jurica and Anamarija, for all of you were unconditionally there to support both me and my family. Even though life always finds a way of tearing us apart we fortunately get back together with smiles and hugs. Now that I've handed in my dissertation, I have no excuse not to visit you guys more regularly; I look forward to our time together and helping each other even more. Thanks to you, I now know that there are people who will answer the phone at all hours. I would also like to mention Hrvoje and Đurđica with whom I simply clicked and thank them for giving me opportunity to be a "naštrigani kum".

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strengthening my faith and bringing both comfort and a way out to my family during difficult times.

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# Summary

This dissertation focuses on uncertainty modeling, robust stability, and performance analysis of coupled multi-input multi-output (MIMO) reduced-order dissipative dynamical systems. Large-scale structural dynamics systems, micro-electro-mechanical systems (MEMS), flexible multi-body dynamics systems (FMBDS), and similar systems are examples of such systems. For modeling and analyzing such systems, spatial discretization techniques (SDMs) such as the finite element method (FEM) are commonly used. With the introduction of supercomputers, finite element analysis (FEA) engineers are now able to conduct analyses with reasonably high fidelity, but at the expense of lengthy computation times and models of a very high order. In contrast, the strong need for real-time robust control - with the advent of digital twin models - necessitates relatively low order models. Herein lies the significance of model order reduction (MOR) strategies. The accuracy of reduced order models (ROM) determines the accuracy of the coupled model. The dissipation theory is fundamental to all aspects of modern robust control, including mathematical (numerical) modeling, analysis, and synthesis. Complex couplings (interconnections) of a large number of mutually interacting dissipative dynamical systems can be efficiently analyzed with tools such as semi-definite programming (SDP), linear matrix inequalities (LMIs), and integral quadratic constraints (IQCs). With appropriate parametrizations and relaxations, this boils down to a convex optimization problem (COP). A viable solution to this COP ensures robust stability and robust performance, which is sometimes referred to as robustness. The behavioral approach, which incorporates zooming, tearing, and linking, was recently proposed as a viable method for analyzing (coupled) dynamical systems. It supports the modeling of coupled dynamical systems in a more natural way, providing an advantage when dealing with coupled dynamical systems' complicated interconnections. While conducting research for this dissertation, the following discoveries were made: (i) a novel concept of multi-scale structure preservation is introduced to emphasize the importance of both global structure preservation and local structure preservation at the subsystem level; (ii) errors introduced into analysis via SDMs and MOR can be modeled as unstructured linear time-invariant dynamic uncertainties; (iii) uncertainty conservatism at the subsystem level can be reduced for a special class of interconnected dissipative dynamical systems; (iv) preserved structure can be used to

account for the dissipative dynamics of the surroundings in analysis and (uncertainty) modeling; and (v) systematic modeling of the uncertainty and model order reduction (MOR) at the level of a subsystem gives both modeling freedom and the ability for obtaining less conservative uncertainties on the level of a subsystem. To generate low-order, robustly stable coupled systems, a new structure-preserving procedure using subsystem partitioning and subsystem MOR through the balanced truncation method (BTM) is presented. Resulting systems are well suited for practical decentralized and distributed robust controller synthesis. The suggested method is also flexible. The suggested design approach allows users to control conservatism reduction or coupled model order. Local structure preservation may be exploited to optimize the dynamical response of large-scale subsystems. Using IQCs and  $\mu$ -analysis, a robustness analysis was conducted on several numerical experiments to support the findings. Multiple configurations of spatially discretized vibration dynamical systems composed of a series of simply supported Euler beams coupled with springs and dampers are investigated. The results suggest that the dissipative behavior of subsystems may be taken into consideration effectively when numerically modeling coupled dynamical systems. The suggested technique is almost certain to produce suboptimal models due to its heuristic approach, yet it represents an appealing design technique that brings up several relevant research questions.

**Keywords:** coupled dynamical systems; structure preserving; robustness analysis; uncertainty modelling; model order reduction; dissipative dynamical systems; second order dynamical systems; spatial discretization error modelling; finite element method;

# Prošireni sažetak

Tema ovog istraživanja je razvoj numeričke metode za analizu spregnutih dinamičkih sustava. Metoda se razvija u okviru teorije automatskog upravljanja i regulacije. Metoda se implementira numerički korištenjem MATLAB-a i odgovarajućih proširenja (eng. toolbox). Spregnuti dinamički sustavi su sustavi sačinjeni od više međusobno povezanih podsustava s višestrukim ulazima i izlazima. Primjeri ovakvih sustava su, između ostalog, sustavi dobiveni u okviru dinamike sustava više deformabilnih tijela (eng. flexible multi-body dynamics systems), mikro-elektro-mehaničke sustavi (eng. micro-electro-mechanical systems), višeagentni robotski sustavi (eng. multi-agent robotic systems) te raspodjela temperature unutar toplinski vodljivih materijala. Svaki podsustav je prostorno diskretiziran korištenjem metode konačnih elemenata (MKE) koji se može zapisati kao linearni dinamički sustav drugog reda. Ubrzanim povećanjem računalne snage, moguće je vršiti analizu sustava dobivenih pomoću MKE, sa gotovo proizvoljno visokom točnošću, na uštrp dužih vremena simulacije uz modele s jako velikim brojem konačnih elemenata.

Za takve sustave, sve je veća potreba za stvaranjem sustava upravljanja i regulacije koji rade efikasno i robustno u realnom vremenu. Moderan pristup ovom problemu je stvaranje vjerne digitalne replike stvarnog sustava, tzv. digitalni blizanac (eng. digital twin) koji zahtjeva da se sustav modelira na fleksibilan način kako bi se dinamika sustava, po potrebi, mogla relativno jednostavno prilagoditi promjenama na stvarnom modelu. Za linearne vremenski invarijantne sustave prvog reda razvijeni su mnogi korisni alati za sintezu sustava upravljanja i regulacije, stoga se takvi modeli najčešće koriste u okviru modernog upravljanja. Pri pretvorbi prostorno diskretiziranih sustava u sustave u prostoru stanja, najveći nedostatak je izrazito veliki broj stanja (reda modela) rezultirajućeg sustava, koji lako naraste do desetaka tisuća ili više. S tako velikim brojem stanja, nemoguće je ili nepraktično pristupiti modeliranju, analizi i sintezi u okviru automatskog upravljanja i regulacije. Iz tog razloga, gotovo neizostavno, koriste se metode redukcije reda modela (RRM). Zbog velike potrebe za RRM, ovo znanstveno područje gotovo je neiscrpno i popraćeno je stalnim i intenzivnim razvojem novih metoda RRM.

Ovdje je bitno za naglasiti sljedeće, iako su modeli u prostoru stanja najzastupljeniji u teoriji upravljanja, također se intenzivno razvijaju i metode upravljanja za sustave

koji nisu zapisani u prostoru stanja. Tu je veliki naglasak stavljen na linearne sustave drugog reda - jer takvi sustavi se direktno dobivaju rješavanjem mnogih fizikalnih problema (kao što je i slučaj sa mehaničkim sustavima koji su prostorno diskretizirani). Ovakva potreba nastala je uz činjenicu da se, između ostalog, nastoje očuvati posebne značajke i struktura podsustava, koji se obično djelomično ili potpuno gube prilikom pretvorbe u sustav prvog reda. Očuvanjem strukture podsustava moguće je dobiti izrazito točne modele niskog reda, pa je ta činjenica popraćena i razvojem RRM za sustave drugog reda. Iako još ne postoji razvijeno robusno sučelje za analizu i sintezu linearnih sustava drugog reda, valja uočiti kako je nedugo (u 2021.), u programskom paketu MATLAB uvedena sveobuhvatna podrška za modeliranje linearnih (mehaničkih) sustava drugog reda. Za modele drugog reda, da se pokazati da su matrice potrebne za opis linearne dinamike izrazito rijetke (eng. sparse), pa se dinamički sustavi vrlo visokog reda u teoriji mogu analizirati efikasno, čak i bez primjene redukcije reda modela. Sve su ovo pokazatelji koji ukazuju na značaj linearnih modela drugog reda, ali i očuvanje strukture modela.

Također je interesantno za uočiti da se mijenja trenutna paradigma razmatranja dinamičkih sustava, a to je modeliranje sustava kao tzv. crne kutije (eng. black-box). Pri ovakvom pristupu, fizikalni (ali i ostali dinamički) sustavi, modeliraju se kao procesor informacija (eng. signal processor). Klasična iterpretacija ulaz-izlaz (eng. input-output) dinamičkih sustava, koja je započela s uvođenjem prijenosnih funkcija (eng. transfer function), polako se odbacuje za modeliranje fizikalnih sustava. Uz ulaz-stanje-izlaz (eng. input-state-output) iterpretaciju koja je trenutno najzastupljenija i koju je uveo R. E. Kálmán u šezdesetim godinama prošlog stolecja, u zadnjih tridesetak godina, razvijaju se i drugi pristupi modeliranju dinamičkih sustava. Izrazito je obećavajuće razmatranje ponašanja (eng. behavioral approach, u nastavku behavioralni pristup) dinamičkog sustava, temeljen na približavanju, raščlanjivanju i spajanju (eng. zooming, tearing and linking) koje je, paralelno sa opisom disipativnih dinamičkih sustava, uveo J. C. Willems. Ova dva koncepta predstavljaju polaznu točku za razvoj novih metoda u svim fazama izučavanja dinamičkih sustava - matematičkom (i numeričkom) modeliranju, analizi i sintezi. Utjecaj koji je Willems sa svojim istraživanjem ostavio na cijelu zajednicu "kontrolaša" (eng. control engineers) biti će uskoro prikazan. Razvojem dvaju spomenutih principa, razvio se i tzv. port-Hamiltonijanski (eng. port-Hamiltonian) način izučavanja i modeliranja fizikalnih dinamičkih sustava. Kod ovog pristupa, dinamički sustav razmatra se kao sustav otvorenog tipa u kojem se preko portova (eng. port) u dinamički sustav uvodi energija koja se jednim dijelom gubi (disipira) van granica sustava, a preostalim dijelom unutar sustava pretvara u druge oblike energije. Pri čemu se, slično kao i kod behavioralnog pristupa, ne proučava izmjena informacija (odnosno signala) među spregnutim sustavima, nego izmjena energije - što svakako ima jasniju iterpretaciju, ali i praktičniju primjenu za fizikalne

sustave. Svi ovi navedeni pristupi omogućavaju bolji uvid u dinamiku fizikalnih sustava jer se temelje na osnovnim zakonitostima fizike i prvim principima (eng. first principles). Uz ovo treba spomenuti još i modeliranje dinamičkih sustava korištenjem računalnog učenja (eng. machine learning) i neuro-neizrazitih-genetskih (eng. neuro-fuzzy-genetic) sustava, koji predstavljaju najmoderniji pristup modeliranju, analizi i sintezi, međutim više su orijentirani ka izrazito nelinearnim dinamičkim problemima. Za prethodno navede pristupe, koji su argumentativno u mnogim slučajevima bolji za modeliranje fizikalnih spregnutih dinamičkih sustava od modela opisanim prostorom stanja, metode upravljanja za takve matematičke modele tek dostižu razinu praktične primjenjivosti.

Od navedenih metoda i pristupa, fokus ove disertacije zadržan je na behavioralnom pristupu i teoriji disipativnosti te do neke mjere na korištenju linearnih sustava drugog reda i popratnim metodama koje se mogu povezati sa trenutno dostupnim metodama u okviru teorije upravljanja. Behavioralni pristup u samoj svojoj definiciji (kroz korake raščlanjivanja i povezivanja) sadrži očuvanje strukture sustava. Važnost očuvanja strukture sustava vidljiv je i kroz značajan interes znanstvenika u posljednjih dvadesetak godina. Treba napomenuti kako se očuvanje strukture kroz literaturu najčešće odnosi ili na strukturu unutar podsustava (u nastavku lokalno očuvanje strukture) ili na strukturu unutar spregnutog sustava (globalno očuvanje strukture). Behavioralni pristup koji je postavio Willems naglašava važnost očuvanja strukture prilikom modeliranja. Međutim, u okviru teorije upravljanja, zanimljivo je za uočiti kako, uz najbolje autorovo znanje, nije izučavan sveobuhvatni pristup očuvanja strukture - i lokalne i globalne strukture istovremeno. Stoga autor ovom prilikom uvodi pojam višeskalnog očuvanja strukture (eng. multi-scale structure preservation) za spregnute dinamičke sustave.

O značaju temeljnih principa teorije disipativnosti koje je postavio Willems 1972., prije točno pedeset godina od danas, govori i činjenica da su upravo u sklopu tog važnog događaja, izdana dva opsežna broja (2 i 3), u svesku 42, od strane IEEE. Svezak 42 i brojevi 2 i 3 su dio *IEEE Control Systems (Magazine)*, a nose naziv *50 Years of Dissipativity, Part I (Part II)*. Autori koji su sudjelovali u izradi navedenih brojeva, također su doprinjeli razvoju mnogih grana teorije upravljanja upravo korištenjem spomenute teorije disipativnosti. Kao istaknute autore, među ostalima, treba navesti C. W. Scherer, S. Weiland, M. Arcaka, L. Grüne, B. Brogliato, T. H. Hughes i R. Patesa, na čijim radovima počiva dobar dio ove disertacije. Osim navedenih znanstvenika, autor bi još izdvojio i A. Megretskog koji je u devedesetima izdao članak (također usko vezan za teoriju disipativnosti) vezan za teoriju integralnih kvadratnih ograničenja (IKO) i time postavio temelje za analitički pristup analizi nesigurnih sustava. IKO omogućavaju razmatranje nesigurnosti odvojeno od nominalnog sustava. IKO defini-

rana su nejednakostima koja se koriste za opisivanje (parcijalno) mogućih kombinacija signala unutar nekog dinamičkog sustava u zatvorenom krugu. Iako IKO predstavljaju izrazito koristan alat u dokazivanju teorema, glavni značaj imaju pri izvođenju algoritama temeljenih na konveksnoj optimizaciji koji se koriste za traženje dozvoljenih rješenja (kandidata) kojima se potvrđuje i dokazuje stabilnost i robusnost nesigurnih dinamičkih sustava.

Osim ovog recentnog događaja, veliku važnost teorije disipativnosti pokazuje i veliki projekt Njemačke Znanstvene Institucije (njem. Deutsche Forschungsgemeinschaft, DFG) pod nazivom *Calm, Smooth and Smart - Novel Approaches for Influencing Vibrations by Means of Deliberately Introduced Dissipation*, koji je započeo 2016., a pod okriljem kojega se odvijaju deseci (uz desetke već završenih) projekata vezanih za disipativnost općenito, ali i mnogi s naglaskom na primjeni u teoriji upravljanja i usmjereni upravo na mehaničke sustave. Među takvim projektima valja izdvojiti projekt, koji je u trenutku pisanja rada još uvijek aktivan, a kojeg vodi P. Benner, pod nazivom *Structure-Preserving Model Reduction for Dissipative Mechanical Systems*. U okviru tog projekta izdan je veliki broj članaka i razvijen veliki broj algoritama vezanih za redukciju reda modela. Kako je vidljivo i iz samog naziva projekta, fokus je na disipativnim sustavima. U okviru projekta, također je razvijen i alat za programski paket MATLAB pod nazivom MORLab koji je izdašno korišten u ovoj disertaciji pri izvođenju numeričkih eksperimenata.

Kako je vidljivo iz dosadašnjeg pregleda, ova disertacija bavi će se sljedećim područjima:

- Metodama prostorne diskretizacije (prvenstveno metoda konačnih elemenata) i teorijom sustava za numeričko modeliranje spregnutih dinamičkih sustava.
- Metodama redukcije reda modela (prvenstveno metode bazirane na redukciji reda modela u prostoru stanja, s naglaskom na metode očuvanja posebnih svojstava sustava, kao što su stabilnost, disipativnost i pasivnost, te metode redukcije očuvanja strukture sustava) u svrhu dobivanja podobnih modela nižeg reda.
- Modeliranjem greške koja nastaje uslijed prostorne diskretizacije i/ili redukcije reda modela (u okviru teorije robusnog upravljanja uz provođenje analize robusnosti nesigurnih sustava korištenjem integralnih kvadratnih ograničenja i strukturirane singularne vrijednosti).

Prostorna diskretizacija parcijalnih diferencijalnih jednadžbi koje opisuju dinamiku sustava, provoditi će se isključivo metodom konačnih elemenata. Valja napomenuti da se i druge metode diskretizacije kao što je metoda konačnih volumena (eng. finite

volume method) i metoda konačnih razlika (eng. finite difference method), također mogu uklopiti u okviru predloženih metoda.

## Hipoteze istraživanja

Hipoteze istraživanja su:

1. Mehanički dinamički sustavi opisani parcijalnim diferencijalnim jednadžbama mogu se modelirati kao niz međusobno povezanih linearnih vremenski-invarijantnih podsustava s nesigurnostima, te je takvim modelom moguće dovoljno točno opisati dinamičko ponašanje ključno za sintezu učinkovitog sustava upravljanja.
2. Moguće je iskoristiti činjenicu da su podsustavi spregnuti da bi se dobilo bolji model nesigurnosti, te time dodatno povećati učinkovitost sustava upravljanja.

## Ciljevi istraživanja

Glavni cilj ovog istraživanja je razviti numeričku metodu za ocjenu točnosti i greške prostorne diskretizacije spregnutih dinamičkih sustava. Uz lokalnu grešku diskretizacije dodatno se uzima u obzir i točnost spregnutog dinamičkog sustava, te se tako dobiva poboljšani model nesigurnosti za spregnute dinamičke sustave. Očekivani cilj, stoga je i unaprijeđenje učinkovitost sustava robustnog upravljanja spregnutih

dinamičkih sustava.

## I. Matematički model spregnutog dinamičkog sustava

Matematičko modeliranje provodi se u okviru teorije sustava i upravljanja. Razmatrani dinamički sustavi diskretizirani metodom konačnih elemenata, rezultiraju linearnim sustavima drugog reda. Dobiveni linearni sustavi drugog reda pretvaraju se u linearne vremenski-invarijantne sustave prvog reda. Za linearne vremenski-invarijantne sustave prvog reda dostupan je velik broj robusnih alata za modeliranje, analizu i sintezu. U okviru ovoga doktorata izdašno se koriste metode redukcije reda modela. Metoda koja predstavlja temelj u razvijenom algoritmu je metoda uravnoteženog skraćivanja (eng. balanced truncation method), stoga je toj metodi posvećena posebna pažnja. Diskretizirani i reducirani modeli povezuju se pomoću algebarskih izraza, pri čemu se zadržava struktura modela. Uz ovako očuvanu strukturu spregnutog sustava, može se pristupiti modeliranju i mijenjanju svakog od podsustava, koje se nakon modeliranja ponovno spaja spregnutom sustavu. Ovakav pristup daje dodatnu slobodu kod modeliranja pri čemu se svaki od podsustava može modelirati na način da zadovoljava lokalne zahtjeve, ali i globalne.

## II. Analiza robusnosti spregnutog disipativnog dinamičkog sustava

Za opisivanje greške diskretizacije i greške redukcije reda modela korišteni su alati iz područja robusnog upravljanja. Netočnost u dinamičkom odzivu i trajektoriji reduciranog sustava u odnosu na nereducirani sustav visokog reda, može se modelirati kao nestrukturirana nesigurnost. U ovom radu korištene su aditivna i multiplikativna nesigurnost. Uz očuvanu globalnu strukturu sustava, moguće je korištenjem linearne frakcijske transformacije (eng. linear fractional transformation) izdvojiti nesigurnosti svih podsustava i na taj način dobiti jednu, blok dijagonalnu nesigurnu



matricu, koja je u povratnoj vezi povezana sa preostalim nominalnim dijelom nesigurnog spregnutog sustava. Za takav sustav, moguće je korištenjem integralnih kvadratnih ograničenja (IKO) i/ili strukturirane singularne vrijednosti (SSV) pristupiti analizi robusnosti. Osim navedenih nesigurnosti koje nastaju uslijed diskretizacije i redukcije reda modela, moguće je istovremeno vršiti analizu robusnosti uslijed drugih tipova nesigurnosti. Za mehaničke sustave to su najčešće parametarske nesigurnosti i određeni tipovi nelinearnosti.

### III. Numerička analiza, numerički eksperimenti i zaključak

Uz prethodno navedene metode, moguće je iskoristiti strukturu sustava kako bi se, osim modelirala nesigurnost za svaki podsustav, također i smanjila njena konzervativnost na nivou podsustava. Konzervativnost nesigurnog modela ima za posljedicu smanjenje performansi robusnog sustava upravljanja. Smanjenje konzervativnosti provodi se na način da se u obzir uzimaju okolni sustavi. Pri tome se koriste saznanja iz teorije disipativnosti kako bi se odbacio dio nesigurnosti koji se zbog same disipativnosti u spregnutom sustavu prigušuje. Predložena numerička metoda provedena je na nizu numeričkih primjera. Primjeri su takvi u kojima se disipativnost manifestira kroz modalno Rayleighovo prigušenje unutar samih diskretiziranih podsustava i kroz prigušne elemente koji spajaju podsustave odnosno Analiza robusnosti potvrđena je korištenjem poznate i u literaturi dostupne metode temeljene na strukturiranim singularnim vrijednostima (eng. structured singular values), odnosno tzv.  $\mu$ -analize, ali i modernijom i sofisticiranijom metodom integralnih kvadratnih ograničenja. Integralnim kvadratnim ograničenjima efikasno se rješavaju i problemi pod utjecajem više vrsta nesigurnosti (eng. mixed uncertainties). Gotovo svi problemi postavljeni na temeljima integralnih kvadratnih ograničenja u konačnici se mogu formulirati kao sustav linearnih matričnih nejednadžbi (eng. linear matrix inequalities), odnosno problem postaje konveksni optimizacijski problem (eng. convex optimization problem). Ovo daje veliku prednost integralnim kvadratnim ograničenjima u analizi robusnosti, jer osim što postoji veliki niz razvijenih i praktičnih alata za rješavanje takve klase problema, dobivena rješenja garantiraju robusnu stabilnost i željenje robusne performanse rezultirajućeg nesigurnog modela niskog reda koji je dobiven spajanjem velikog broja

prostorno diskretiziranih podsustava niskog reda.

## Ostvareni znanstveni doprinosi

Na osnovu prikazanih numeričkih rezultata, diskusije i zaključaka, potvrđene su i prethodno postavljene hipoteze istraživanja.

Osim potvrđenih hipoteza, iz ove doktorske disertacije proizašli su i sljedeći znanstveni doprinosi:

- Razvijen je originalni koncept modeliranja dinamičkih sustava s očuvanjem strukture na više razina, koji osigurava očuvanje strukture modela dinamičkog sustava na globalnoj razini te na razini podsustava.
- Pokazano je kako se greške modeliranja sustava koje su posljedica prostorne diskretizacije i posljedica redukcije reda modela mogu učinkovito modelirati kao nestrukturirane linearne vremenski-invarijantne dinamičke nesigurnosti.
- Pokazano je kako se konzervativizam u modeliranju greške na nivou podsustava može učinkovito smanjiti za posebnu klasu spregnutih disipativnih dinamičkih sustava.
- Pokazano je kako se očuvanje strukture modela dinamičkog sustava može učinkovito iskoristiti u analizi i modeliranju (nesigurnog) dinamičkog sustava koji se nalazi u sprezi s disipativnim sustavom ili sustavima koji predstavljaju njegovu okolinu.
- Razvijena je metoda sustavnog modeliranja nesigurnosti i redukcije reda modela na razini podsustava koja omogućava slobodu modeliranja i manju konzervativnost modela nesigurnosti na nivou podsustava.

**Ključne riječi:** spregnuti dinamički sustavi; očuvanje strukture; analiza robustnosti; modeliranje nesigurnosti; redukcija reda modela; disipativni dinamički sustavi; integralna kvadratna ograničenja; dinamički sustavi drugog reda; modeliranje greške prostorne diskretizacije; metoda konačnih elemenata;

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## CHAPTER 1

# Introduction

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## 1.1 Motivation

Mathematical and numerical modeling is an essential tool in many technological disciplines. Complex dynamical systems are composed of multiple-input, multiple-output (MIMO) subsystems, each of which can be driven by a distinct physical principle. Each subsystem can be formally characterized using ordinary differential equations (ODE), differential-algebraic equations (DAE), and partial differential equations (PDE), and then linked together using additional algebraic connections. Large-scale structural dynamics systems, micro-electromechanical systems (MEMS), flexible multi-body dynamics systems (FMBS), multi-agent robotic systems (MARS), very large system integrated (VLSI) chip design, smart structures, and networks of embedded systems (NES) are examples of such systems. These and many other related systems belong to a class of spatially distributed dynamical systems that have acquired prominence in a variety of scientific and technological domains, including control theory applications.

There is a rising demand for real-time robust control, real-time system monitoring, and real-time adaptation of a model to reflect changes in a real physical system for the previously identified coupled dynamical systems. This can be accomplished by creating a replica of a physical system and a real-time information exchange interface between the replica and the physical system, such that both evolve in tandem over

time. A digital twin modelling is becoming increasingly popular [1]. An effective digital twin model should roughly satisfy the following requirements: (i) be adaptable, i.e. easy to change, (ii) have relatively low order, i.e. the real time simulation should require a small amount of computer resources, (iii) be accurate, i.e. have a dynamical behavior similar to that of a physical system, and (iv) be robust, i.e. capable of handling unknown external disturbances and changes within the model. A notable example of building such digital twin models is being carried out as part of an ongoing research project led by Z. Tonković, called Protection of Structural Integrity in Energy and Transport Industries, which is dealing with structural health monitoring of wind turbines, among other things.

One method for making models adaptive is to preserve the important properties of each subsystem in the coupled model. Among other things, these could be model parameters (e.g., mass, damping, stiffness, and so on) that can be altered if necessary, and then other properties should be kept intact for such a changed model - for example, subsystem stability, passivity, reciprocity, and so on. Couplings (interconnections) between subsystems can also be preserved, allowing a subsystem to be readily altered, replaced, or eliminated as necessary.

Designing a low order model that is understood and can be dealt with efficiently - a linear and possibly time-invariant model - is a control engineer's goal as old as control and systems theory itself. On the contrary, the models resulting from spatial discretization are often large-scale and of high order. To address this in a practical manner, one can employ model order reduction methods. However, not all model order reduction methods satisfy the first requirement, leaving us with only a subset of methods to deal with - those that do preserve important properties.

The third criteria is a little more difficult to cope with. To begin with, obtaining an accurate model is a difficulty in and of itself - how much of the dynamics of the real system should be included in the model for it to be an accurate description of the real model? Second, a trade-off must be made because the first two conditions are fundamentally opposed to this one. To deal with this, one must frequently prioritize which aspects of the subsystem to focus on, such as preserving only some of the properties or only important parameters, preserving only necessary couplings, employing a model order reduction method that results in a model with high fidelity in a specific frequency range or yielding a model with good correlation in time domain simulations, and so on.

However, the fourth requirement provides considerable relief from all of the preceding requirements. If the coupled dynamical system is modeled within the robust control theory framework, it is possible to robustify the model by introducing uncertainty - a set of dynamical responses that were not included in the original model but are

anticipated the model might exhibit. This uncertainty set should be as (mathematically) simple as possible while also not being overly conservative - that is, it should not include too much of the excluded dynamics. Uncertainty can be thought of as a safety factor: too little of it has no effect, too much of it causes the resulting system to be excessively conservative.

Because the eventual goal of mathematical modeling is typically merely a first step in a much bigger engineering problem, it should be approached both systematically and pragmatically. According to the details provided thus far, numerical modeling of coupled dynamical systems proves to be a challenging task. This dissertation is an attempt to tackle at least some of the challenges acknowledged above. The main focus of this dissertation will be on uncertainty modeling and the uncertainty conservatism reduction for coupled dynamical systems comprised of uncertain low order discretized subsystems. The resulting coupled systems should be robustly stable and meet the specified robust performance criteria, and as such, be suitable for practical robust control.

## 1.2 Literature review

In order to model, analyze, and synthesize such systems in an acceptable amount of time, spatial (and occasionally temporal) discretization is necessary [2]. The body of knowledge in the field of spatial discretization methods, as well as the availability of robust software solutions for finite element analysis (FEA), reflect the field's maturity [3], [4], [5]. Most common spatial discretization methods (SD) are the finite difference method (FDM), finite volume method (FVM), and finite element method (FEM). Continuous spatially distributed interconnected dynamical systems are relatively easily discretized using FEM. The main concept in the application of SD methods is to achieve solution convergence by constructing a sequence of finer discretizations (meshes, grids) [6]. The idea that finer discretizations result in more accurate solutions is extensively documented in the literature [3]. Regardless of mesh density, discretization error (DE) is unavoidable; the only question is when and in what amount it is acceptable. Traditional DE estimation approaches could be used in some situations [7]. Many cases and finite element types have well-known error limitations for structural analysis [8]. There are a number of examples where the error bounds are known for dynamical and vibration problems, but as can be seen, those are fairly problem particular [9], [10]. As a result, conventional DE estimation bounds have been demonstrated to be difficult or impractical to execute for the challenges addressed in this dissertation - numerical modeling of coupled dynamical systems in a system and control theoretic setting.

From the standpoint of a FEA engineer, the question is not if a mesh that offers an accurate solution to the problem will be found, but when it will be found. That is, arbitrarily accurate finite element models can be designed with a relatively large number of nodes (elements) at the expense of longer calculation times. These FEM models with a large number of nodes ultimately become high-order dynamical systems. By contrast, when a control engineer considers the identical problem (i.e., a spatially distributed dynamical system discretized using FEM), the shift in priority occurs quite early. Due to typically amount of the available computing resources for an average controller, in order for a control engineer to model and synthesize an effective controller, a low-order dynamical system must be considered. As a result, the following question inspires the study of control of spatially distributed and discretized systems:

"When is discretization of spatially distributed systems *good enough* for control?"

Jones has addressed this question and shown in his work [11], through the usage of the  $\nu$ -gap metric developed by Vinnicombe [12], that a level of discretization (and consequently the order of a resulting dynamical system), for small to middle sized systems, can be chosen such that the robustly synthesised controller for the original system, will have good performance for the lower discretization (order) system as well. The paper by Jones also considered a rather specific system and the obtained results did not scale as well for large-scale problems such as discretized mechanical systems with multiple inputs and outputs. Part of the dissertation, as it will be shown in the sequel, will still rely on the cognition made by Jones. Although not directly for modeling but rather for evaluation and verification of to be presented method.

Evidently, dealing with extremely large order dynamical systems in a system and control theoretic framework is rather peculiar. To remedy this, one should incorporate model order reduction theory, methods and algorithms [13]. Model order reduction (MOR) plays an essential role in the mathematical modelling of large-scale dynamical systems [14]. Similar to spatial discretization, MOR renders many real world problems analysable in a reasonable time [15]. Using reduced order models (ROM) does come at the expense of introducing MOR errors (MORE) into the analysis. As one can observe, there are similarities with discretization orders and chosen orders for ROMs. Low order discretized models introduce significant amount of error into the analysis and low order ROM introduce significant amount of error into the analysis as well. Again, interesting motivational question arise:

"When are reduced order models of spatially discretized distributed systems *good enough* for control?"





Many influential and regarded academics (mainly from control theory, but also from other fields) have expressed their admiration and provided important insights into how the theory of dissipative systems assisted them in achieving their research goals. A body of high-quality knowledge can be discovered in these two issues. Authors, whose works are tightly related to this dissertation, who contributed to making these two (indeed special) issues, alphabetically ordered, are: M. Arcak [21], B. Brogliato [22], L. Grüne [23], D. J. Hill [24], T. H. Hughes [25], R. Pates [26], C. W. Scherer [27] and S. Weiland [28]. The works listed for each author, although directly related to dissipativity, and thus referenced here, will be explained in more details later throughout the dissertation in appropriate places.

Another indicator of the importance of theory of dissipativity in general, but also highly associated control theory and mechanical systems is the research project, with dozens of highly influential smaller projects, supported by German Research Foundation (ger. Deutsche Forschungsgemeinschaft, DFG) under the name *Calm, Smooth and Smart - Novel Approaches for Influencing Vibrations by Means of Deliberately Introduced Dissipation* [29]. Perhaps, one of the major developmental breakthroughs, that highly influenced and is relevant to this dissertation, was done with P. Benner and his team. Their research resulted in a numerous of problem specific MOR techniques based on theory of dissipation and special subsystem properties preservation. More details on these MOR techniques are given later.

So far, the investigation has been restricted to a single large-scale spatially discretized system. The treatment of a single discretized spatially distributed large-scale dynamical system or one produced by coupling numerous subsystems was the same. That is, even though one discusses a complex interconnected systems, it was still viewed as a single large-scale system. This type of modeling and analysis does have advantages, as the black-box approach can be particularly useful for analysis of specific systems. However, as will be demonstrated in the following section, it is critical to preserve these interconnections and evaluate such a system as a dynamical system comprised of subsystems that can interact with one another through their interconnections. This is known as a structure-preserving framework, and it is an essential part of this dissertation. The importance of this will be revised in a sequel.

From now on, the focus will be shifted more on how to systematically model each subsystem to obtain overall accurate and appropriate coupled system. Coupled systems are often also called composite or interconnected as well. The coupled systems under consideration will be the ones that are comprised (mostly) of dissipative dynamical subsystems. Once more, it is interesting to note that Willems had a significant interest in this area of research as well [30], [31], [32], [33], [34], with some of these papers having high influence in the field. The following is a central idea from Willems' work

that is related to both dissipativity and coupled systems and is heavily used in this dissertation: "It is shown that dissipative systems which are interconnected via a neutral interconnection constraint define a new dissipative dynamical system and that the sum of the storage functions of the individual subsystems is a storage function for the interconnected system" [17].

Perhaps what had a bigger impact, not only on the field of control and systems, but on other fields as well, was another important concept that he introduced, a that is a concept of behavioral approach to dynamical systems [35], [36], [37], [38], [39]. The behavioral approach (BA) is tightly intertwined with the study and analysis of coupled dynamical systems [40], [41], [33], willems2009, [42]. Although the BA was not employed directly or to its full extent, the dissertation author believes it had a significant impact on both his understanding of dissipative systems and later assumptions about the dissipative nature of linked dissipative systems. The main principle behind BA is to not evaluate the system's dynamics as an input/output response (i.e classical transfer function). Willems emphasized that Kálmán's work and the introduction of input/state/output (or today's classical state-space representation) was a significant step toward unifying representation and modeling of dynamical systems. Due to other works of Willems even the input/state/output is lacking and could be approached in a more simpler yet more powerful manner. The intricate details and theory behind the BA is out of scope for this dissertation and the interested readers are referred to the previously cited papers and these [43], [44].

One can now arrive at yet another interpretation of coupled systems that Willems introduced. It deals with a three step modelling procedure called tearing, zooming and linking (also referred to as juxtaposition and interconnection). The concepts presented in [45], [46], [47] and [48] serve a purpose of another central point for this dissertation, that are nicely summarized as: "In general it is not possible to speak about the energy that flows in and out of a mechanical system along a set of terminals. In order for one to be able to speak about the energy transferred, certain conditions on the forces that act on these terminals have to be satisfied, namely, the sum of the forces that act on the terminals has to be zero. One can refer to a set of terminals that satisfies this condition as a 'port'." [48]. Two most important concepts from the previous citation are the energy transfer and the choice of the couplings (connections, or ports and terminals) between the subsystems. For now only, thing to keep in mind is that energy (in the papers also regarded as systemic property) is not a local property but rather involves the couplings as it is transferred accros the terminals inside the system. This will be elaborated in greater detail later in the dissertation.

Until now, only modeling details regarding the known parts of the coupled systems were considered. Let us now tackle the unknowns - the unknown DE and MORE.

In a system and control theoretic framework, the way to deal with the unknowns is through the robust control theory [49], [50], [51], [52]. Robust control environment demands modeling of the uncertainty - an unknown or a trouble-making part of the system. Uncertainty modeling is a subtle yet powerful way to include the deviations in dynamical behavior between the real model (a physical system, a chemical process or a plant) and a modeled uncertain dynamical system [53]. In this definition the real physical plant is considered, for which the dynamical behavior is unavailable to the control engineer - this in turns requires a leap of faith to be taken when modeling of uncertain dynamical system is done [49]. When modeling uncertainty in this dissertation, a fact that an arbitrarily fine discretization will eventually lead to an arbitrarily good approximation of the dynamical behavior will be used, such that the 'real' plant will be the finest available discretization. In this scenario, when a reduced order discretized dynamical system is compared to its 'real' representation, a typical divergence in dynamical response in higher frequency and overall dynamics similar to those of unmodeled and discarded dynamics can be observed. It is well known in the literature that these kind of deviations can be modeled using unstructured uncertainties (UU) [49], [51], [52]. Typical uncertainty models used to describe the UU (which will also be used herein) are additive uncertainty (AU) and multiplicative uncertainties (MU). The implementation of AU and MU depends highly on the choice of frequency weighted filters (or usually called just weights) used [49], [51]. The importance of weights, in general, is recognized as an important topic for decades since the choice of a weight design directly influences the robustness and performance of a controller.

Now that the uncertainty (coming from DE and MORE) modeled for each subsystem was carried out and an uncertain coupled system obtained, the crucial step remains. This includes proving robust stability and robust performance - often shortened to just robustness analysis. A well received and extensively used tool, readily available in MATLAB as part of the Robust Control Toolbox™[54], that relies on calculation of the so called structured singular value is the so called  $\mu$ -tools introduced by Doyle in 1982 [55]. It was extended to include complex structured singular value [56] and in general relies heavily on the linear fractional transformation (LFT) (also known as the Redheffer star product [57]). LFT unifies the frequency-domain and state-space methods and is used to 'pull-out' the uncertainties ('unknown' part) from the uncertain system. This in turn results in the nominal ('known') system being connected in feedback with a block-diagonal (and usually complex) matrix that includes all the uncertainties.

In the 1997, yet another powerful tool for robustness analysis was introduced, namely the integral quadratic constraints (IQCs) by Megretski and Ratzner [58]. Introduction

of the IQCs has given the study of uncertainties a broader and analytical analysis approach. Indeed, IQC allow separate analysis of a nominal and the trouble making part, which was proven to introduce analysis flexibility when compared to classical methods [58], [59], [60]. Together with dissipativity theory, research development in recent years, yielded practical tools based on the powerful framework of IQC, suitable for usage in both frequency and time domains [61], [62], [63], [64], [27]. Veenman et al. recently bridged the gap between IQC theory and its practical usage and wrote a summary with examples of practical applicability of the IQC, that can be found [65]. What makes the usage of IQC particularly attractive, is that they essentially boil down to convex optimization problems, linear matrix inequalities (LMI) [60], and semi-definite programs (SDP) [66], [67]. These can be efficiently solved with many of the available tools. IQCs might soon reach the deserved engineering recognition and start being used as often as the popular  $\mu$ -tools. Robust stability and performance analysis - robustness analysis - via IQC becomes a viable and practical option by using recently developed "IQClab: A new IQC based toolbox for robustness analysis and control design" [68]. Together with LMI parsers such as LMILab [69],[70] (available in MATLAB Robust Control Toolbox™[71], YALMIP [72], [73] or CVX [74], [75], and SDP/LMI solvers such as mincx (part of MATLAB LMILab), SeDuMi [76], SDPT3 [77], and MOSEK [78]. According to the presented literature, the rapid development of (robust) control theory, both in theoretical and practical aspects, is firmly based on the theory of dissipativity and elegant mathematical proofs that, for numerous practical cases, reduce to a convex optimization problem (COP), such as LMIs and SDPs. The significance of convex optimization within control theory cannot be overstated, since it would require at least as much text to explain as the famous dissipation theory. Here, convex optimization will be viewed as a practical and powerful tool for confirming the modeling results. In conclusion, one can hope that the reader is left with the impression that the focus of this dissertation is on modeling, but convex optimization may be more relevant in the realm of robust analysis and synthesis. Let us continue the investigation.

The two previously posed questions can now be stated more accurately and in combination with all of the previously stated important ideas. This dissertation's prime motivating question is as follows:

**How may uncertainties in coupled systems made of spatially discretized, reduced order, and dissipative dynamical subsystems be modeled in order to construct a low order robustly stable coupled uncertain system suited for practical robust control?**

With all of the important ingredients laid out previously, let us expand a bit on each of them to find specific tools needed to answer to this question.

First let us consider on how to preserve the structure of dynamical system. Even though the importance of structure preservation is evident from the given literature it seems quite unusual that, other than Willems who indirectly speaks about structure preservation on both a level of a subsystem (from now on called local structure preservation) as well as on a level of a coupled system (from now on called global structure preservation), there aren't any, to the best of authors knowledge, unified frameworks that explicitly state the importance of this is neither how to approach it for some specific modeling problem. The author of this dissertation would like to take this opportunity to coin a name for this framework - a multi-scale structure preservation framework. Some of the examples that state the importance of such a framework and in what way it may be exploited will be given later, together with other main results of this dissertation.

The local structure preservation is usually tightly linked with MOR and other approximation techniques. Important for the content of this dissertation are the structure preserving techniques reflected on the preservation of the mass, stiffness and damping matrices structure for the second-order linear (often called mechanical) dynamical systems [79], [80], [81], [82], [83], [84], [85]. While second-order systems can readily be converted to the first-order linear descriptor state-space models, there is a lack of robustness analysis tools - for example IQCs are not suitable for such systems. Some interesting results will still be shown for these systems, using  $\mu$ -analysis, based on [86]. On the other hand, usage of linear second-order systems, although presented attractive, are still heavily hampered by the lack ready-made and robust solutions for both analysis as well as synthesis within the current control and system theoretic framework. In this dissertation, the results are mainly derived using first-order linear dynamical system in its explicit form (i.e  $E = I$ ). By doing so, the local structure will be lost, but justified in the discussion. There are also local structure preservation techniques for first-order explicit systems, but with the previous statement, there is no practical usage of such methods in this case.

The conclusions and key outcomes of this dissertation, on the other hand, will heavily rely on the results from the body of knowledge available with regard to global structure preservation of coupled (interconnected) systems. The following works are related to either general structure preservation, MOR reduction of coupled systems [87], [88], [89], control of interconnected systems [90], [91], stability of dissipative couplings [92], [93], [94], [21], decentralized and distributed control of coupled systems [95], [96], [97], [98] or combination of any [90], [99], [100]. Most of these works were used to familiarize with the topic of coupled dynamical systems and to recognize the important properties a resulting model must have in order to be suitable for further steps in control (i.e robust and/or decentralized and/or distributed controller

synthesis). Details from some of these works are used while deriving the main results and thus mentioned at these appropriate places as well.

Besides the structure preservation, it is usually important to preserve some special subsystem properties in a ROM. Perhaps the most important property to preserve is the stability and this fact is clearly stated for each MOR technique (i.e. does it preserve stability or not)[101]. Since this dissertation only deals with LTI stable dynamical systems, only such MOR techniques were utilized. Sometimes, it is also important for a MOR technique to preserve some other vital properties. This is especially important in decentralized and distributed controller synthesis, because decentralized/distributed controller synthesis can be carried out easier and the controller performance readily improved. Some of these properties, and works related to these are, passivity preservation [102], [103], [104], [105], [106], phase preservation [107] and frequency response preservation in a limited frequency range. A work that plays an important role in this dissertation is dealing with the strictly dissipative state-space representation of second-order (mechanical) systems [108].

Few books that served for building fundamental knowledge about dissipative (coupled) systems, (robust) stability are by Gupta [109], Siljak [110], Zečević and Siljak [111], Mohhammadpour and Grigoriadis [112], Molyan [113], Arcak et al. [114] and Brogliato et al. [115].

Few last word should be given to other concurrent approaches. State-of-the-art opponents to the approach presented in this dissertation - i.e. modeling of spatially discretized and reduced order coupled dynamical systems in control and system theoretic framework - are listed next. As a first opponent, author of this thesis would pick a behavioral approach. A lot was already said about behavioral approach, but to summarize, behavioral approach offers interpretation of the physics and physical system on a rather simplistic level, based on first principles and not the exchange of information but rather the exchange of energy between the system and its environment (through the dissipation theory) and throughout the system with systemic energy flow (or as power) [43], [44]. The second opponent is taken to be port-Hamiltonian dynamical system modeling. Port-Hamiltonian dynamical system modeling is continuation of the classical Hamiltonian approach, but directly influenced by behavioral approach (the main reason author first picked behavioral approach). As of date, the port-Hamiltonian approach is arguably superior to behavioral approach - mainly due to larger scientific community and longer history (due to classical Hamiltonian approach) [116]. The main advantage over behavioral approach lies in the rapid development of control schemes, specifically tailored towards control of port-Hamiltonian systems [117]. The uncertainties can also be dealt with within this framework [118]. Similar to behavioral approach it is based on the direct notion of dissipativity together with energy storage

and energy transfer to and from the coupled systems, but between the subsystems as well [119]. Another beauty of the port-Hamiltonian approach is that it can be represented as an input-state-output system with some (and in some cases all) properties preserved [120], [121]. Port-Hamiltonian systems tolerate model order reduction with structure and parameter preservation [83].

### 1.3 Dissertation objective and hypotheses

The objective of this research is to develop a numerical method for modeling coupled dynamical systems. The coupled dynamical systems are influenced by uncertainties occurring from spatial discretization and model order reduction. The uncertainties will be modeled on the level of a subsystem in a newly developed structure-preserving scheme. This preserved structure together with subsystems' unique properties - namely the dissipativity - will be used in conjunction to reduce uncertainty conservatism. The resulting uncertain coupled system will be of low order, robustly stable and meet the specified robust performance criteria. As a result, improved performance of robust control system for coupled dynamical systems is achieved.

To this day, the following hypotheses for this dissertation are:

- Mechanical dynamical systems described with partial differential equations can be modeled as series of coupled linear time-invariant subsystems with uncertainties and by obtaining this model it is possible to describe, with sufficient accuracy, dynamic behavior crucial for synthesis of efficient control system.
- The fact that each subsystem is coupled with other subsystems may be used to obtain better uncertainty model and thus additionally improve efficiency of control system.

### 1.4 Dissertation outline

In this first chapter the main motivation is given, possible difficulties and challenges when approaching the problem of numerical modeling of coupled dynamical systems are highlighted and a thorough literature overview, including both seminal works and body-of-knowledge is given. State-of-the-art in the field of system theoretic and robust control theory approach to modeling of coupled dynamical systems, both highly related to the dissertation as well as on overview of possible other approaches to modeling of coupled dynamical systems, was laid out.

In the second chapter, necessary mathematical concept will be given, followed by the definition of dynamical systems used in this dissertation — specifically, first-order and second-order linear (and time-invariant) dynamical systems are defined. The definition of coupled dynamical systems is also given with the emphasis on the structure preserving modeling of coupled dynamical systems. After that a brief overview of model order reduction methods is given with the emphasis on balanced truncation method. A details on how to apply balanced truncation method within structure preserving framework is presented as well.

The third chapter focuses on uncertainty modeling and robustness analysis. The errors resulting from spatial discretization and model order reduction methods can be considered as modeling error or neglected higher order dynamics. These types of errors can be efficiently modeled using unstructured uncertainties, the first part of this chapter deals exactly with this — how are unstructured uncertainties modeled and the necessary ingredients for obtaining uncertain models is given. Next, an important mathematical machinery, required for setting up the problem for the robustness analysis — linear fractional transformation — is presented and explained. With the model being represented through a linear fractional representation in the following part, first an introduction to the robustness analysis is given. Then the robustness analysis using integral quadratic constraints is explained, together with the mathematical machinery behind it.

Fourth chapter deals with numerical modeling of coupled dynamical systems. Two scenarios are represented. First scenario serves as a motivation and introduction into modeling of dissipative coupled dynamical systems. It is based on non structure-preserving approach and highlights the possibility of exploiting dissipativity and intricate couplings of the system in reducing the uncertainty conservatism using classical methods without emphasis on robustness analysis. It was shown that with rather simple representation of the uncertainty using multiplicative uncertainty model, that it is possible to reduce the conservatism of the uncertainty that is result of spatial discretization. Then, the structure-preserving approach to uncertainty modeling and uncertainty conservatism reduction is given. A detailed design procedure for approaching the numerical modeling of dissipative coupled dynamical systems is given. Highlights of the benefits of structure-preservation and the flexibility of the proposed design method, as well as possible shortcomings are given.

In the fifth chapter, a numerical experiments on a more practical physical example is given. A series of simply supported Euler beams coupled by springs and dampers are considered in a structure preserving manner. On a number of different cases, the importance of spatial distribution and position, the amount of dissipativity and other details, is exploited to draw conclusions on the efficacy, practicability and limitations of



the proposed method. In the discussion section of this chapter, previously highlighted aspects are further investigated.

In the final section the conclusions of this dissertation are presented. Main scientific contributions of the dissertation are outlined as well as the limitations of the used methods and procedures.

## CHAPTER 2

# Mathematical modelling of coupled dynamical systems

### CHAPTER CONTENTS

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## 2.1 Notation and preliminary mathematical concepts

Let  $\mathcal{L}_2$  denote the space of vector-valued square integrable functions with a fixed number of components (depending on the context) and that it represents signals with finite energy.  $\mathcal{L}_2$  is equipped with standard inner product and norm. A space of real rational and proper transfer matrices  $\mathcal{RH}_\infty$  without poles in the closed right half plane that have a finite  $\mathcal{H}_\infty$ -norm and a space  $\mathcal{RH}_\infty$  without poles on the imaginary axis is also needed. Operators (or dynamical systems) are maps  $G : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ , that take any input  $w \in \mathcal{L}_2$  into the output  $z \in \mathcal{L}_2$ . For linear  $G$  one can denote the output as  $z = Gw$ . One can denote the induced  $\mathcal{L}_2$ -gain norm as  $\|G\|$ —which for LTI systems equals the  $\mathcal{H}_\infty$  norm,  $\|G\|_{\mathcal{H}_\infty}$ ).

## 2.2 First-order and second-order linear dynamical systems

Let us consider  $k$  interconnected structural spatially distributed dynamical subsystems discretized with finite elements. For each subsystem, governing equations of motion

written in second-order form [3, 2] are

$$\mathcal{M}_j \ddot{q}_j(t) + \mathcal{P}_j \dot{q}_j(t) + \mathcal{K}_j q_j(t) = F_j(t), \quad (2.1)$$

where  $q_j(t)$  represents the generalized coordinates at nodes (i.e., displacements and rotations),  $\dot{q}_j(t)$  represents the generalized velocities at nodes (i.e., linear and angular velocities),  $\ddot{q}_j(t)$  represents the generalized accelerations at nodes (i.e., linear and angular accelerations), while  $\mathcal{M}_j$  is the mass matrix,  $\mathcal{P}_j$  is the damping matrix,  $\mathcal{K}_j$  is the stiffness matrix, and  $F_j(t)$  is the vector of applied external nodal forces [3, 2, 122] at the  $j$ -th subsystem with number of systems being  $j = 1, \dots, k$ . The above equation can be written in the first-order descriptor state-space form [122], a first order system can be as well represented as

$$\begin{aligned} \underbrace{\begin{bmatrix} \mathcal{M}_j & 0 \\ 0 & I \end{bmatrix}}_{E_j} \dot{x}_j(t) &= \underbrace{\begin{bmatrix} -\mathcal{P}_j & -\mathcal{K}_j \\ I & 0 \end{bmatrix}}_{A_{jD}} x_j(t) + \underbrace{\begin{bmatrix} B_{j1} \\ 0 \end{bmatrix}}_{B_{jD}} w_j(t), \\ z_j(t) &= \underbrace{\begin{bmatrix} C_{j1} & 0 \\ 0 & C_{j1} \end{bmatrix}}_{C_j} x_j(t), \end{aligned} \quad (2.2)$$

where  $B_{j1}$  is the matrix defining node locations at which external inputs (i.e., forces) are applied,  $C_{j1}$  is the matrix defining node locations at which the generalized velocities and generalized coordinates are measured. It can be shown that for nonsingular  $E_j$  this system can be represented and used in its state-space form as

$$\begin{aligned} \dot{x}_j(t) &= A_j x_j(t) + B_j w_j(t), \\ z_j(t) &= C_j x_j(t), \end{aligned} \quad (2.3)$$

where  $A_j = E_j^{-1} A_{jD}$  is the state matrix,  $B_j = E_j^{-1} B_{jD}$  is the input matrix, and  $C_j$  is the output matrix.

A starting point for obtaining equations 1 2 3 4 was the spatial discretization of a mechanical subsystem The focus in this dissertation is on FEM, but arguably, similar conclusions with regards to discretization errors can be made for FVM and FDM as well. Also, the proposed methods and procedures in the sequel should in theory be applicable for these discretizations methods as well.

## 2.3 Coupled dynamical systems

Consider a system of  $k$  interconnected continuous LTI state-space subsystems defined with Equation (2.3) that are interconnected through the relation

$$\begin{aligned} w_j(t) &= K_{j1}z_1(t) + \dots + K_{jk}z_k(t) + H_j w(t), \\ z(t) &= R_1z_1(t) + \dots + R_kz_k(t), \quad j = 1, \dots, k. \end{aligned} \quad (2.4)$$

where  $u_j(t)$  are internal inputs and  $z_j(t)$  are internal outputs,  $K_{jl} \in \mathbb{R}^{m_j, p_l}$ ,  $H_j \in \mathbb{R}^{m_j, m}$  and  $R_j \in \mathbb{R}^{p, p_j}$  are interconnection matrices, while  $w(t)$  is an external input and  $z(t)$  is an external output of a subsystem. Interconnected systems represented by Equations (2.3) and (2.4) are often also called *interconnected systems* or *composite systems* [102, 87].

Let  $n = n_1 + \dots + n_k$ ,  $p_0 = p_1 + \dots + p_k$ , and  $m_0 = m_1 + \dots + m_k$ . Now consider the coupling matrices

$$\begin{aligned} R &= [R_1, \dots, R_k] \in \mathbb{R}^{p, p_0}, \\ H &= [H_1^T, \dots, H_k^T]^T \in \mathbb{R}^{m_0, m}, \\ K &= [K_{j,l}]_{j,l=1}^k \in \mathbb{R}^{m_0, p_0}, \end{aligned} \quad (2.5)$$

together with the block diagonal matrices

$$\begin{aligned} A &= \text{diag}(A_1, \dots, A_k) \in \mathbb{R}^{n, n}, \\ B &= \text{diag}(B_1, \dots, B_k) \in \mathbb{R}^{n, m_0}, \\ C &= \text{diag}(C_1, \dots, C_k) \in \mathbb{R}^{p_0, n}. \end{aligned} \quad (2.6)$$

Then a state-space representation of the interconnected system is given by

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \mathcal{A} &= A + BKC \in \mathbb{R}^{n, n}, \\ \mathcal{B} &= BH \in \mathbb{R}^{n, m}, \\ \mathcal{C} &= RC \in \mathbb{R}^{p, n}. \end{aligned} \quad (2.8)$$

There are attempts in the literature to exploit the structure of the system through these interconnection matrices. However, in the case of this dissertation, author relied on a robust approach that is already implemented in MATLAB. The function called

**connect** allows one to make interconnected model by defining names of inputs and outputs for each subsystem. The connections are defined through algebraic relations. Algebraic relations can be created with the so called summing blocks with a function named **sumblk**. As it will be seen in the sequel, systems that are interconnected in this way are suitable in the step of uncertainty modelling and refinement.

## 2.4 Balanced truncation method

The balanced truncation method (BTM) is one of the most studied reduction techniques [14], [79], [103]. It was developed primarily for the reduction of the state space models, which are arguably the most suitable models for most numerical applications [60], [66], [67]. The main disadvantage of this method is that generalized Lyapunov equations have to be solved, which can be computationally demanding for high order systems [14], [79]. There is development on low rank approximations to the solutions of the Lyapunov matrix equations that make balanced truncation attractive for large-scale problems as well [102, 79].

Since each subsystem can be governed by completely different physical laws and act in different spaces and time scales, applying the BTM directly on the interconnected system completely destroys the structure (or interconnections) [123]. Changing the interconnected model even slightly requires recalculation and reduction of the entire model once again. Opting for BTM for each subsystem can thus be advantageous. Specific reduction criteria can be applied to each subsystem while preserving the structure and properties of the resulting reduced interconnected system as well [106], [124], [125], [126], [127], [84].

The main paradigm in applying the BTM can be explained in the frequency domain [14, 102]. The model reduction problem can be formulated as follows. For a given subsystem  $G_j(s) = C_j(sI - A_j)^{-1}B_j$  find an approximation  $\tilde{G}_j(s) = \tilde{C}_j(sI - \tilde{A}_j)^{-1}\tilde{B}_j$  where  $\tilde{A}_j \in \mathbb{R}^{l_j \times l_j}$  for some  $l_j \ll n_j$  such that  $\|\tilde{G}_j - G_j\|_{\mathbb{H}_\infty}$  is small.

The BTM method is related to the controllability Gramian  $\mathcal{P}_j$  and observability Gramian  $\mathcal{Q}_j$  that are, for each subsystem, unique symmetric, positive semi-definite solutions of the generalized Lyapunov equations

$$A_j \mathcal{P}_j + \mathcal{P}_j A_j^T + B_j B_j^T = 0, \quad (2.9)$$

$$A_j^T \mathcal{Q}_j + \mathcal{Q}_j A_j + C_j^T C_j = 0. \quad (2.10)$$

A system is called balanced if  $\mathcal{P}_j = \mathcal{Q}_j = \text{diag}(\sigma_{j,1}, \dots, \sigma_{j,n_j})$  where  $\sigma_{j,i} = \sqrt{\lambda_{j,i}(\mathcal{P}_j \mathcal{Q}_j)}$  are the Hankel singular values of the subsystem described by the Equation (2.3). The general idea behind BTM is to transform the subsystem (2.3) into a balanced form

and to truncate the states that correspond to the small Hankel singular values (HSV). Balancing and truncation can be performed in a numerically efficient way with the following algorithm [102], [79].

Although Algorithm 1 can be relatively easily implemented, the author would like to suggest using already available routines in the software of choice, as those are probably more robust solutions—especially when reducing the system of extremely high orders [107]. As such, Algorithm 1 still serves a purpose of explaining the balanced truncation method and main reasons to opt for such a method in the proposed procedure—which one can restate here, are—(i) Hankel singular values have to be calculated only once and a series of different orders of reduced order models can be obtained at no additional numerical cost, and (ii) stability of the reduced order system is preserved.

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**Algorithm 1:** *Generalized square root balanced truncation method.*

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- 1 For the subsystem defined with the Equation (2.3) with the transfer function calculated as  $G_j(s) = C_j (sI - A_j)^{-1} B_j$  compute the reduced order system.
- 2 Compute the (lower) Cholesky factors  $L_{\mathcal{P}_j}$  and  $L_{\mathcal{Q}_j}$  of the Gramians  $\mathcal{P}_j = L_{\mathcal{P}_j} L_{\mathcal{P}_j}^T$  and  $\mathcal{Q}_j = L_{\mathcal{Q}_j} L_{\mathcal{Q}_j}^T$ , that satisfy the generalized Lyapunov Equations (2.9) and (2.10).
- 3 Compute the singular value decomposition

$$L_{\mathcal{P}_j}^T L_{\mathcal{Q}_j} = [U_{j,1}, U_{j,2}] \begin{bmatrix} \Sigma_{j,1} & 0 \\ 0 & \Sigma_{j,2} \end{bmatrix} [V_{j,1}, V_{j,2}]^T,$$

where  $[U_{j,1}, U_{j,2}]$  and  $[V_{j,1}, V_{j,2}]$  have orthonormal columns,  $\Sigma_{j,1} = \text{diag}(\sigma_{j,1}, \dots, \sigma_{j,l_j})$  and  $\Sigma_{j,2} = \text{diag}(\sigma_{j,l_j+1}, \dots, \sigma_{j,r_j})$  with  $r = \text{rank}(L_{\mathcal{P}_j}^T L_{\mathcal{Q}_j})$ .

- 4 Compute the reduced order system

$$\begin{aligned} \dot{\tilde{x}}_j(t) &= \tilde{A}_j \tilde{x}_j(t) + \tilde{B}_j \tilde{w}_j(t), \\ \tilde{z}_j(t) &= \tilde{C}_j \tilde{x}_j(t), \end{aligned} \tag{2.11}$$

with  $\tilde{A}_j = \mathcal{W}_j^T A_j \mathcal{T}_j$ ,  $\tilde{B}_j = \mathcal{W}_j^T B_j$  and  $\tilde{C}_j = C_j \mathcal{T}_j$ , where  $\mathcal{W}_j = L_{\mathcal{Q}_j} V_{j,1} \Sigma_{j,1}^{-1/2}$  and  $\mathcal{T}_j = L_{\mathcal{P}_j} U_{j,1} \Sigma_{j,1}^{-1/2}$ .

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## CHAPTER 3

# Uncertainty modelling and robustness analysis

### CHAPTER CONTENTS

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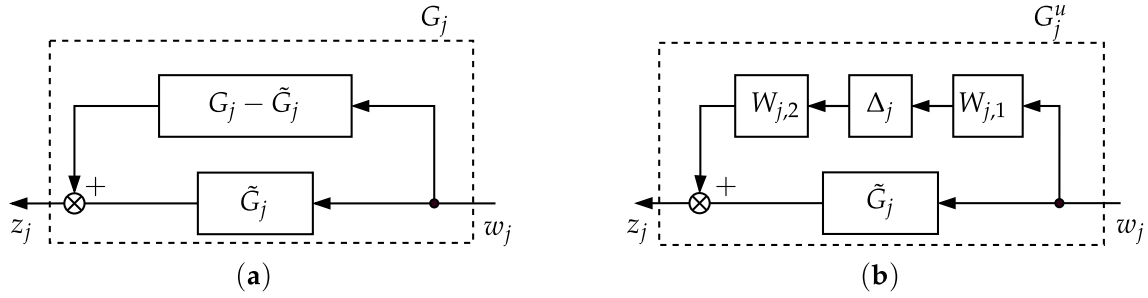
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## 3.1 Unstructured uncertainties

In order to capture the unknown effect that spatial discretization and MOR have on the resulting discretized and reduced order subsystems, the uncertainty is introduced. The type of uncertainty that is introduced into the analysis with spatial discretization and MOR can be considered as a modelling error [128, 60]. This type of uncertainty can be modelled using input- or output-multiplicative uncertainty or additive uncertainty [66], [60], [128].

Due to the nature of the problem, having to model absolute gaps between the nominal and the uncertain model, and in order to preserve the structure of an interconnected system, an additive uncertainty model is used [128]. The absolute error between the original model and the reduced order discretized model can be calculated as  $G_j - \tilde{G}_j$ , and adding it back to the reduced order discretized model  $\tilde{G}_j$ , clearly results in a nominal model  $G_j$ . The absolute error and this type of addition is represented in Figure 3.1a.



**Figure 3.1:** Modelling the absolute error between the original model and the reduced order discretized model as an uncertain dynamical system with additive uncertainty: (a) Adding the absolute error  $G_j - \tilde{G}_j$  to the reduced order discretized model  $\tilde{G}_j$ . (b) Representing the absolute error as an additive uncertainty  $W_{j,2}\Delta_j W_{j,1}$ .

The uncertainty can be introduced to the system as shown in Figure 3.1b. This uncertain system can be written by replacing the absolute error  $G_j - \tilde{G}_j$  with an appropriate uncertainty model as

$$G_j^u = \tilde{G}_j + W_{j,2}\Delta_j W_{j,1}, \quad (3.1)$$

where  $G_j^u$  is the resulting uncertain subsystem,  $\tilde{G}_j$  represents a discretized and/or reduced order subsystem,  $W_{j,1}$  and  $W_{j,2}$  are assumed to be stable frequency weights, and  $\|\Delta_j\|_{\mathcal{H}_\infty} \leq 1$  is the uncertainty.

## 3.2 Linear fractional transformation

Let us now consider the system represented as in Figure 3.2. The frequency weights  $W_{j,1}$  and  $W_{j,2}$  can be incorporated into the nominal system, and together with the nominal part of the system are now defined as  $\Gamma$ . This augmentation, shown in Figure 3.2, can be performed with linear fractional transformation (LFT) as

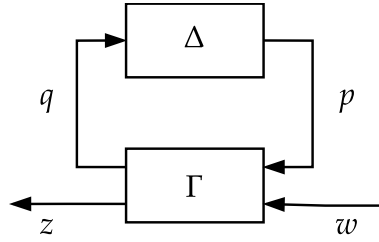
$$\begin{bmatrix} q \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \Gamma_{qp} & \Gamma_{qw} \\ \Gamma_{zp} & \Gamma_{zw} \end{bmatrix}}_{\Gamma} \begin{bmatrix} p \\ w \end{bmatrix}, \quad p = \Delta(q), \quad (3.2)$$

which is often denoted using the *Redheffer star product* [128], or the  $\star$  operator, as  $z = (\Delta \star \Gamma)w$ , where

$$\Delta \star \Gamma := \Gamma_{zw} + \Gamma_{zp}\Delta(I - \Gamma_{qp}\Delta)^{-1}\Gamma_{qw},$$

and which is assumed to be well-posed for all  $\Delta \in \mathbf{\Delta}$ , where  $\mathbf{\Delta}$  represents some set in which  $\Delta$  can take values, that identifies the nature and structure of uncertainties (for more details see [65]). Here  $\Gamma \in \mathcal{RH}_\infty$  is a stable LTI system where  $p \rightarrow q$  represents the uncertainty channel and  $w \rightarrow z$  represents the performance channel.





**Figure 3.2:** Standard feedback interconnection for robustness analysis.

With simple rearranging of the terms represented in Equation (3.1), and rewriting it as Equation (3.2), the expression for the LFT of an uncertain system with additive uncertainty reads as

$$\begin{bmatrix} q_j \\ z_j \end{bmatrix} = \begin{bmatrix} 0 & W_{j,1} \\ W_{j,2} & \tilde{G}_j \end{bmatrix} \begin{bmatrix} p_j \\ w_j \end{bmatrix}. \quad (3.3)$$

### 3.3 IQC analysis

Integral quadratic constraints (IQC) [58] represents an established unifying framework for the analysis of both robust stability and robust performance of uncertain systems. IQC essentially allow the analysis of a rich class of uncertainties by detaching the uncertainty  $\Delta$  from the nominal part  $\Gamma$ . The IQC analysis can be carried out without the explicit knowledge of the uncertainty  $\Delta$  but rather with limited overall information of the energy transfer concerning its input-output properties. As shown in [58], this usually applies to a much larger set of uncertain operators that belong to a certain class  $\Delta$ . The IQC analysis can be carried out in the frequency domain by finding solutions to frequency domain inequality (FDI) on a fixed grid of finite frequencies (thus making this semi-infinite problem). It was proven that this FDI can be turned into a finite convex feasibility problem with suitably parametrizing the multipliers [65], [60], [63], [27]. After applying the so called Kalman–Yakubovich–Popov (KYP) lemma, the IQC problem can also be analysed in time domain [65], [60], [63], [27]. A summary in details of steps required to transform this rather abstract concept into a computationally tractable problem can be found in [65], while here only some key points will be outlined. To illustrate this concept, let us examine the system as shown in Figure 3.2. First, the behaviour on the channel  $p \rightarrow q$  representing the uncertainty channel is studied.

Two signals  $p \in \mathcal{L}_2$  and  $q \in \mathcal{L}_2$  are said to satisfy the IQC defined by the multiplier  $\Pi$  [65] if

$$\mathcal{I}(\Pi, q, p) := \left\langle \begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \right\rangle \geq 0, \quad (3.4)$$

where  $\Pi$  is said to satisfy  $\Pi = \Pi^* \in \mathcal{RL}_\infty$ . With  $p = \Delta(q)$  IQC now reads

$$\mathcal{I}(\Pi, q, \Delta(q)) \geq 0 \quad \forall q \in \mathcal{L}_2 \quad (3.5)$$

it becomes possible to capture useful properties of uncertainties  $\Delta \in \mathbf{\Delta}$  and describe its input-output energy transfer relations. In order to carry out a practical robust stability analysis, it is usual to construct a whole family of multipliers  $\mathbf{\Pi} \subset \mathcal{RL}_\infty$  such that Equation (3.4) holds for all  $\Pi \in \mathbf{\Pi} \subset \mathcal{RL}_\infty$  and for all  $\Delta \in \mathbf{\Delta}$ .

In complete analogy, let us also examine the behaviour of the channel  $w \rightarrow z$  on which one wants to impose certain performance criteria, such that two signals  $w \in \mathcal{L}_2$  and  $z \in \mathcal{L}_2$  are said to satisfy the IQC defined by the multiplier  $\Pi_p$  [65] if

$$\mathcal{I}_p(\Pi_p, z, w) := \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} \Pi_{p11} & \Pi_{p12} \\ \Pi_{p12}^* & \Pi_{p22} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle \geq -\epsilon \|w\|^2, \quad (3.6)$$

where  $\epsilon > 0$  is used to capture strict a version of the imposed performance specification. In order to carry out a practical robust performance analysis, it is usual to construct a whole family of multipliers  $\mathbf{\Pi}_p \subset \mathcal{RL}_\infty$  such that Equation (3.6) holds for all  $\Pi_p \in \mathbf{\Pi}_p \subset \mathcal{RL}_\infty$ , while the  $\Pi_p$  is confined to the set

$$\mathbf{\Pi}_p \subset \{\Pi_p \in \mathcal{RL}_\infty : \Pi_{p11} \succcurlyeq 0\}$$

where the  $\Pi_{p11} \succcurlyeq 0$  is required in order to perform robust stability and performance at the same time, which one will often call just robustness analysis [65].

Let us now state the central theorem, through a frequency domain inequality, as obtained in [58]. In all the FDIs one can use short notation where  $G = G(i\omega)$ ,  $\Pi = \Pi(i\omega)$  and  $\Pi_p = \Pi_p(i\omega)$ .

**Theorem 3.1** Assume that

1. for all  $\tau \in [0, 1]$  the interconnection defined with Equation (3.2), with omitted performance channels  $w$  and  $z$ , is well posed for  $\Delta$  replaced by  $\tau\Delta$ ;
2. for all  $\tau \in [0, 1]$  and some  $\Pi = \Pi^* \in \mathcal{RL}_\infty$ , the IQC defined with Equation (3.4) is satisfied for  $\Delta$  replaced by  $\tau\Delta$ ;
3. the following FDI is satisfied:

$$\begin{pmatrix} \Gamma \\ I \end{pmatrix}^* \Pi \begin{pmatrix} \Gamma \\ I \end{pmatrix} \prec 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (3.7)$$

Then the interconnection defined by Equation (3.2), with omitted performance channels  $w$  and  $z$ , is stable.

With that in mind, the IQC for robustness analysis can be represented through a FDI as outlined in [65] which is an extension of Theorem 3.1.

**Corollary 3.2** Assume that

1. for all  $\Delta \in \mathbf{\Delta}$  the interconnection defined by Equation (3.2), with omitted performance channels  $w$  and  $z$ , is well posed;
2. for all  $\Delta \in \mathbf{\Delta}$  and for all  $\Pi \in \mathbf{\Pi}$  the IQC defined by Equation (3.4) is satisfied.

Then the interconnection defined with Equation (3.2) is robustly stable and robust performance on the channel  $w \rightarrow z$  is guaranteed if there exists a  $\Pi \in \mathbf{\Pi}$  and  $\Pi_p \in \mathbf{\Pi}_p$  such that

$$\begin{pmatrix} \Gamma_{qp} & \Gamma_{qw} \\ I & 0 \\ \Gamma_{zp} & \Gamma_{zw} \\ 0 & I \end{pmatrix}^* \begin{pmatrix} \Pi & 0 \\ 0 & \Pi_p \end{pmatrix} \begin{pmatrix} \Gamma_{qp} & \Gamma_{qw} \\ I & 0 \\ \Gamma_{zp} & \Gamma_{zw} \\ 0 & I \end{pmatrix} \prec 0. \quad (3.8)$$

The proof can be found in Appendix A in [65]. Robustness analysis, according to Corollary 3.2, simply boils down to checking if there exists a  $\Pi \in \mathbf{\Pi}$  and  $\Pi_p \in \mathbf{\Pi}_p$  such that FDI defined with Equation (3.7) holds true, under the assumptions that both conditions in Corollary 3.2 are met. If so, then the robust performance is achieved and the uncertain system is robustly stable.

To render the presented IQC framework for robustness analysis computationally tractable, a suitable parametrization of  $\mathbf{\Pi}$  and  $\mathbf{\Pi}_p$  is required, such that Equation (3.8) results in a linear constraint on some set of unknown variables [65, 63]. This can be achieved if the families of multipliers are parametrized as

$$\mathbf{\Pi} = \{ \Psi^* P \Psi : P \in \mathbf{P} \}, \quad (3.9)$$

$$\mathbf{\Pi}_p = \{ \Psi_p^* P_p \Psi_p : P_p \in \mathbf{P}_p \}, \quad (3.10)$$

with an LMtable sets  $\mathbf{P}$  and  $\mathbf{P}_p$  of real symmetric matrices  $P \in \mathbf{S}$  and  $P_p \in \mathbf{S}$ , respectively, and some fixed and typically tall transfer matrices  $\Psi \in \mathcal{RH}_\infty$  and  $\Psi_p \in \mathcal{RH}_\infty$ . A set is LMtable if it can be represented as the feasible set of an LMI constraint [60, 67]. Now the robustness analysis can be characterized as follows.

**Corollary 3.3** Assume that

1. for all  $\Delta \in \mathbf{\Delta}$  the interconnection defined with Equation (3.2) is well posed;
2. for all  $\Delta \in \mathbf{\Delta}$  and for all  $P \in \mathbf{P}$  the IQC defined with Equation (3.4) is satisfied with  $\Pi = \Psi^* P \Psi$ .

Then the interconnection defined with Equation (3.2) is robustly stable and robust performance on the channel  $w \rightarrow z$  is guaranteed if there exists a  $P \in \mathbf{P}$  and  $P_p \in \mathbf{P}_p$  such that

$$\begin{pmatrix} \Gamma_{qp} & \Gamma_{qw} \\ I & 0 \\ \Gamma_{zp} & \Gamma_{zw} \\ 0 & I \end{pmatrix}^* \begin{pmatrix} \Psi^* P \Psi & 0 \\ 0 & \Psi_p^* P_p \Psi_p \end{pmatrix} \begin{pmatrix} \Gamma_{qp} & \Gamma_{qw} \\ I & 0 \\ \Gamma_{zp} & \Gamma_{zw} \\ 0 & I \end{pmatrix} \prec 0. \quad (3.11)$$

The obtained FDI defined with Equation (3.11) is affine in the matrix variables  $P$  and  $P_p$ . Therefore, if both sets  $\mathbf{P}$  and  $\mathbf{P}_p$  are LMIable, one obtain a semi-infinite convex robustness analysis feasibility test of the system represented with Equation (3.2) [65, 63]. This FDI needs to hold true for all  $\omega \in \mathbb{R} \cup \{\infty\}$ , or on the fixed grid of properly distributed frequencies. To avoid this, at the cost of the increased computation, it is possible to satisfy Equation (3.11) for all frequencies  $\omega \in \mathbb{R} \cup \{\infty\}$  by using the Kalman–Yakubovich–Popov (KYP) lemma (often called positive-real and bounded-real lemma).

**Lemma 3.4** Let  $P \in \mathbf{S}$  and let  $\Gamma \in \mathcal{RL}_\infty$  admit the realization  $(A_\Gamma, B_\Gamma, C_\Gamma, D_\Gamma)$  with  $A_\Gamma \in \mathbb{R}$  and  $\text{eig}(A_\Gamma) \cap \mathbb{C}^0 = \emptyset$ . The following two statements are equivalent:

1.  $\Gamma^* P \Gamma \prec 0$ .
2. There exists a matrix  $X \in \mathbf{S}$  such that

$$\begin{pmatrix} I & 0 \\ A_\Gamma & B_\Gamma \\ C_\Gamma & D_\Gamma \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} I & 0 \\ A_\Gamma & B_\Gamma \\ C_\Gamma & D_\Gamma \end{pmatrix} \prec 0. \quad (3.12)$$

The corresponding equivalence persists to hold for

- non-strict inequalities, if, in addition, the pair  $(A_\Gamma, B_\Gamma)$  is controllable,
- equalities, if, in addition,  $A_\Gamma$  is Hurwitz and the pair  $(A_\Gamma, B_\Gamma)$  is controllable.

Proof and the details of the KYP lemma can be found in [65, 60, 63] and the references therein. Now the FDI defined with the Equation (3.11) can be checked numerically.

For this purpose let us introduce the following realization

$$\begin{pmatrix} \Psi & 0 \\ 0 & \Psi_p \end{pmatrix} \begin{pmatrix} \Gamma_{qp} & \Gamma_{qw} \\ I & 0 \\ \Gamma_{zp} & \Gamma_{zw} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_R & B_{R1} & B_{R2} \\ C_{R1} & D_{R11} & D_{R12} \\ C_{R2} & D_{R21} & D_{R22} \end{pmatrix} \quad (3.13)$$

where  $A_R \in \mathbb{R}$ ,  $\text{eig}(A_R) \subset \mathbb{C}^-$ , and state the following.

**Corollary 3.5** Assume that

1. for all  $\Delta \in \mathbf{\Delta}$  the interconnection defined by Equation (3.2) is well posed;
2. for all  $\Delta \in \mathbf{\Delta}$  and for all  $P \in \mathbf{P}$  the IQC defined by Equation (3.4) is satisfied with  $\Pi = \Psi^* P \Psi$ . Then the interconnection defined with the Equation (3.2) is robustly stable and robust performance on the channel  $w \rightarrow z$  is guaranteed, if there exist  $X \in \mathbf{S}$ ,  $P \in \mathbf{P}$  and  $P_p \in \mathbf{P}_p$  such that

$$\begin{pmatrix} I & 0 & 0 \\ A_R & B_{R1} & B_{R2} \\ C_{R1} & D_{R11} & D_{R12} \\ C_{R2} & D_{R21} & D_{R22} \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P_p \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ A_R & B_{R1} & B_{R2} \\ C_{R1} & D_{R11} & D_{R12} \\ C_{R2} & D_{R21} & D_{R22} \end{pmatrix} \begin{pmatrix} I & 0 \\ A_\Gamma & B_\Gamma \\ C_\Gamma & D_\Gamma \end{pmatrix} \prec 0. \quad (3.14)$$

Hence, if  $\mathbf{P}$  and  $\mathbf{P}_p$  are LMlible sets, a finite dimensional convex feasibility test for robustness analysis is obtained.

To complete the analysis via IQC, let us also formulate multiplier classes suitable for uncertainties dealt with in this paper as well as a multiplier class suitable for performance cost in this paper. Throughout the paper a basis-function  $\psi_v \in \mathcal{RH}_\infty^{(v+1) \times 1}$  that is fixed and has a McMillan degree of  $v$  is used, to obtain inner approximations (i.e., subsets) of the multipliers defined with the Equations (3.9) and (3.10), defined as

$$\psi_v(i\omega) := \left( 1 \quad \frac{1}{(i\omega-\rho)} \quad \frac{1}{(i\omega-\rho)^2} \quad \cdots \quad \frac{1}{(i\omega-\rho)^v} \right)^T \quad (3.15)$$

with the minimal state-space realization

$$\psi_v = \left( \begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array} \right) = \left( \begin{array}{ccccc|c} \rho & 0 & \dots & \dots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & 1 & \rho & 0 \\ \hline & & & & & 1 \\ & & & & I_v & 0 \end{array} \right) \quad (3.16)$$

where  $\rho < 0$  represents the location of the pole and  $v \in \mathbb{N}$ . By changing  $\rho$  over line search with for example  $v \in \{1, 2, 3, 4, 5\}$  allows for dynamics in the multipliers and gives freedom in search for the feasible solution of Equation (3.14). This implies that the parametrization using such basis function is sufficiently rich to approximate general sets of multipliers. For other basis-functions that can be used, readers are referred to [65, 68]).

Unstructured uncertainties belong to a class of an uncertain LTI dynamics for which it can be said that  $\Delta$  is confined to a set of LTI dynamic full-block uncertainties

$$\Delta_{\text{lti,dyn,fb}} := \{\Delta \in \mathcal{H}_\infty : \|\Delta\|_\infty \leq 1\}. \quad (3.17)$$

Uncertainties of this form can be captured with the following multiplier class. For all the  $\Delta \in \Delta_{\text{lti,dyn,fb}}$  the IQC (3.5) holds with

$$\Psi^* P \Psi := \begin{pmatrix} \psi_v \otimes I & 0 \\ 0 & \psi_v \otimes I \end{pmatrix}^* \begin{pmatrix} P_{11} \otimes I & 0 \\ 0 & -P_{11} \otimes I \end{pmatrix} \begin{pmatrix} \psi_v \otimes I & 0 \\ 0 & \psi_v \otimes I \end{pmatrix}, \quad (3.18)$$

if

$$\psi_v^* P_{11} \psi_v \geq 0. \quad (3.19)$$

Here,  $P_{11} \in \mathbb{S}^{v+1}$  is a free matrix variable and  $\psi_v$  is a fixed basis-function as defined with Equation (3.15). By using the basis-function as defined with Equation (3.16) in its state-space form, one can infer that Equation (3.19) is equivalent to the existence of some matrix  $X_v \in \mathbb{S}^v$  such that the following LMI holds true

$$\begin{pmatrix} I & 0 \\ A_v & B_v \\ C_v & D_v \end{pmatrix}^T \begin{pmatrix} 0 & X_v & 0 \\ X_v & 0 & 0 \\ 0 & 0 & P_{11} \end{pmatrix} \begin{pmatrix} I & 0 \\ A_v & B_v \\ C_v & D_v \end{pmatrix} \succcurlyeq 0. \quad (3.20)$$

As it will be stated in the next chapter, one also needs a performance criterion on the channel  $w \rightarrow z$  for the reduced order interconnected system to be kept as close as possible to the original (i.e., unreduced) interconnected system. The class of performance criteria, suitable for this matter, can be expressed in terms of induced  $\mathcal{L}_2$ -gain. Consider the stable system  $\Gamma_{zw} \in \mathcal{RH}_\infty$  and suppose there exists some  $\gamma > 0$  and some small  $\epsilon > 0$  such that for all trajectories of  $z = \Gamma_{zw}w$ , with  $w \in \mathcal{L}_2$ , the performance IQC defined with Equation (3.6) is satisfied with

$$\Psi_p^* P_p \Psi_p := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \gamma^{-1} I & 0 \\ 0 & -\gamma I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (3.21)$$

Then the induced  $\mathcal{L}_2$ -gain from  $w$  to  $z$  is less than  $\gamma > 0$ . Equation (3.21) can be linearised using Schur-complement [65]. In our case, the interest is on the best achievable induced  $\mathcal{L}_2$ -gain of the uncertain system defined with the Equation (3.2), while guaranteeing robust stability for all  $\Delta \in \Delta_{\text{lti,dyn,fb}}$ . This can be achieved by applying Corollary 3.5 with Equations (3.18), (3.19) and (3.21).

### 3.4 Robustness analysis under mixed uncertainties

Besides tackling both robust stability and robust performance at the same time, another useful aspect of the IQC framework advantage is the possibility to perform robust stability of the interconnected systems where each of the subsystem can be affected by multiple uncertainties. A standard IQC framework can be expanded relatively with  $\Gamma$  and  $\Delta$  being

$$\begin{pmatrix} \Gamma_{qp} & \Gamma_{qw} \\ \Gamma_{zp} & \Gamma_{zw} \end{pmatrix} := \begin{pmatrix} \Gamma_{q_1 p_1} & \cdots & \Gamma_{q_1 p_k} & \Gamma_{q_1 w} \\ \vdots & \ddots & \vdots & \vdots \\ \Gamma_{q_k p_1} & \cdots & \Gamma_{q_k p_k} & \Gamma_{q_k w} \\ \Gamma_{z p_1} & \cdots & \Gamma_{z p_k} & \Gamma_{z w} \end{pmatrix} \quad (3.22)$$

and

$$\Delta(q) := \text{col}(\Delta_1(q_1), \dots, \Delta_k(q_k)) \quad (3.23)$$

where subsystems  $j = 1, \dots, k$  have linear fractional dependency on  $\Delta_1, \dots, \Delta_k$  and  $\Delta_j$  take their values in the sets  $\Delta_j$  [65]. Then Equation (3.2) needs to be well-posed for all  $\Delta_j \in \Delta_j$ , where individual  $\Delta_i$  blocks satisfy the IQC  $\mathcal{I}(\Pi_i, q_i, \Delta_i(q_i)) \geq 0, \forall q_i \in \mathcal{L}_2$ , but also the composite IQC  $\mathcal{I}(\Pi, q, \Delta(q)) \geq 0, \forall q \in \mathcal{L}_2$  holds, with  $q := \text{col}(q_1, \dots, q_k)$

and

$$\Pi = \begin{pmatrix} \text{diag}(\Pi_{1,11}, \dots, \Pi_{k,11}) & \text{diag}(\Pi_{1,12}, \dots, \Pi_{k,12}) \\ \text{diag}(\Pi_{1,12}^*, \dots, \Pi_{k,12}^*) & \text{diag}(\Pi_{1,22}, \dots, \Pi_{k,22}) \end{pmatrix}.$$



## CHAPTER 4

# Numerical modeling of dissipative coupled dynamical systems

### CHAPTER CONTENTS

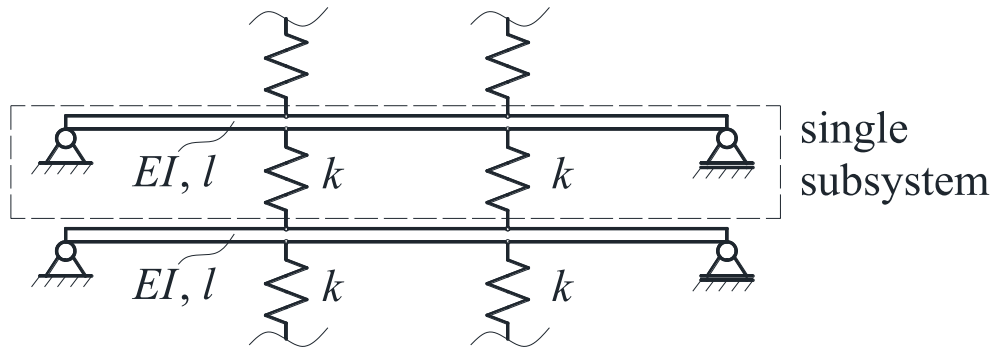
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## 4.1 Uncertainty conservatism reduction without structure preservation

In this motivational example, let's consider vibrations of a mechanical system comprised of a finite number of simply supported Euler beams. The beams are mutually interacting and are coupled with springs as shown in Figure 4.1. Each beam has a circular cross-section and is defined by the geometric and material properties given in Table 4.1, where  $l$  is length,  $d$  is the diameter of the cross-section,  $\rho$  is mass density,  $E$  is the modulus of elasticity and  $\nu$  is Poisson's coefficient. Each beam is connected to two neighboring beams at one third and two thirds of their lengths with a spring of stiffness  $k$ , also given in Table 4.1.



**Figure 4.1:** Spatially distributed mechanical system comprised of simply supported Euler beams coupled with springs

**Table 4.1**  
Subsystem properties

$l$ m	$d$ mm <sup>2</sup>	$\rho$ kg/m <sup>3</sup>	$E$ GPa	$k$ N/m
0.6	1	7800	$210 \times 10^9$	500

Each beam is modeled using finite element method and for each beam a series of successively finer discretizations are made. A beam is discretized with 3, 6, 12, 24, 48 and 96 finite elements. For each discretization, a proportional Rayleigh damping matrix is constructed such that modal damping ratio is approximately 0.05 for the first 6 beam vibration modes. Discretized beams are then connected with springs at specified points in order to obtain a series of successively finer dynamical models. Discretized subsystem models consisting of a single beam and springs is then converted into a linear time-invariant (LTI) descriptor state space (DSS) model with two inputs and two outputs. Inputs are displacements of two spring ends and outputs are beam deflections at one third and two thirds of its length. Based on the obtained LTI models one constructs a nominal model and a series of uncertainty models corresponding to each discretization, as follows. Nominal subsystem model is obtained by reducing the 96 finite element model, that is, nominal model is the reduced most accurate LTI model available. The reduction of the model is obtained by means of a balanced truncation. The criteria for reducing the model is removing all states that have Hankel singular values less than a prescribed value of  $10^{-2}$ . This results in reducing the 576 number of states for the full model (obtained from 96 finite elements) to 19 states for the reduced model. The reduced model is suitable to be used as a nominal model for constructing unstructured uncertainty weights as it will be shown in the next section. By using model reduction, the efficiency of numerical modeling is improved. Perhaps the most widely used tools in robust control design are weights. Weights allow a control designer to capture the system uncertainties and to determine over what frequency

ranges the performance is desired. Weights can be viewed as a way to tune the control design to achieve a desired level of performance and robustness without having to explicitly consider the system stability. “The design of a robust and high-performance control system relies heavily on the choice of weights used in the design. [129]”. The analysis is started by considering uncertainty caused by unmodeled dynamics and assume that the real plant is obtained by combining the nominal plant  $P_{sys}$  with a transfer function of the kind  $W_1\Delta W_2$ . Where  $\Delta$  is uncertainty block and  $W_1$  and  $W_2$  are frequency weights that adjust the amount of uncertainty at each frequency. Such kind of uncertainty is denoted as unstructured. The procedure of constructing unstructured uncertainty models for the presented case is as follows.

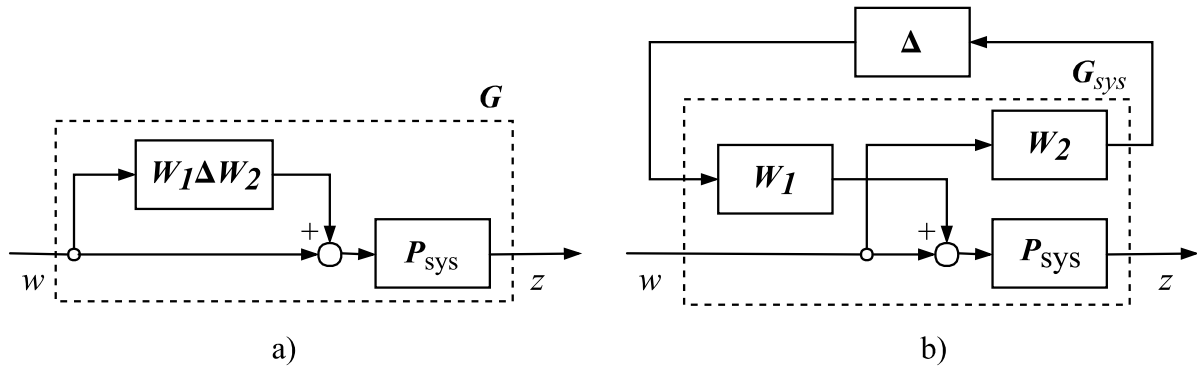
#### 4.1.1 Unstructured uncertainties for a single subsystem

For each discretization level, input multiplicative error models are constructed. An input multiplicative error model consists of an unstructured uncertainty block  $\Delta$  of a unit-gain and a diagonal, stable, minimum-phase shaping frequency weights  $W_1$  and  $W_2$ . The described input multiplicative model is schematically shown in Figure 4.2a. At each input denoted by  $w$ , a frequency weight is constructed such that the response from  $w$ -th input to  $z$ -th output covers the relative error between nominal model and discretized model. Throughout the procedure an order (number of states) of frequency weights has been kept constant at each input. In order to make the size of the uncertainty model small, a low order of 8 for diagonal entries in weight  $W_1$  at each input, whereas weight  $W_2$  is a scalar of magnitude 1, is used.  $P_{sys}$  is an interconnection transfer matrix function such that a linear fractional transformation with uncertainty block yields input multiplicative uncertainty model. Input multiplicative uncertainty [49] is expressed by

$$G = P_{sys}(I + W_1\Delta W_2) \quad (4.1)$$

where  $G$  is an uncertain system obtained by using the input multiplicative uncertainty model to alter the response of a system  $P_{sys}$ .

In a different study made in [130] the authors obtained similar results by using an additive uncertainty model. In a similar fashion they constructed  $\Delta$  with norm equal to 1 but used a scalar weight  $W$ . Additive uncertainty model is usually used to model an absolute error or to model a modest difference in responses of two systems. Scalar weights are used to cover the net response from each input to each output of the error between the nominal model and the discretized model. Uncertainty modeling using scalar weights is more conservative than using a weight at each input of the system. A less conservative uncertainty model can be obtained by using diagonal weights.

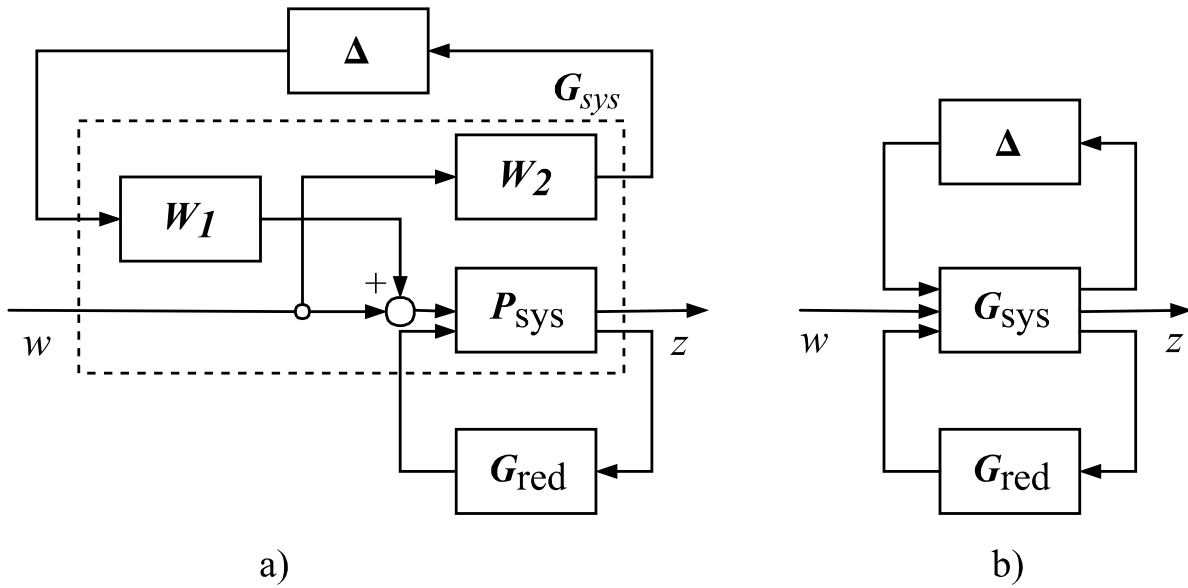


**Figure 4.2:** Input multiplicative uncertainty model: a) a subsystem, b) linear fractional representation of a subsystem

Author stresses that, by different choice of  $P_{sys}$ , different uncertainty models can be used in a straightforward fashion. In robust control theory one wants to obtain a system in linear fractional transformation (LFT). For constructing LFT it is important to separate a normed uncertainty part  $\Delta$  and connect it in feedback to a nominal LTI system. For input multiplicative error model, uncertainty separation can be done as it is shown in Figure 4.2b. By doing so one can now consider a system  $G_{sys}$  that has an uncertainty block  $\Delta$  connected in feedback. In the next section one can show how to refine the uncertainty model by examining a coupled system.

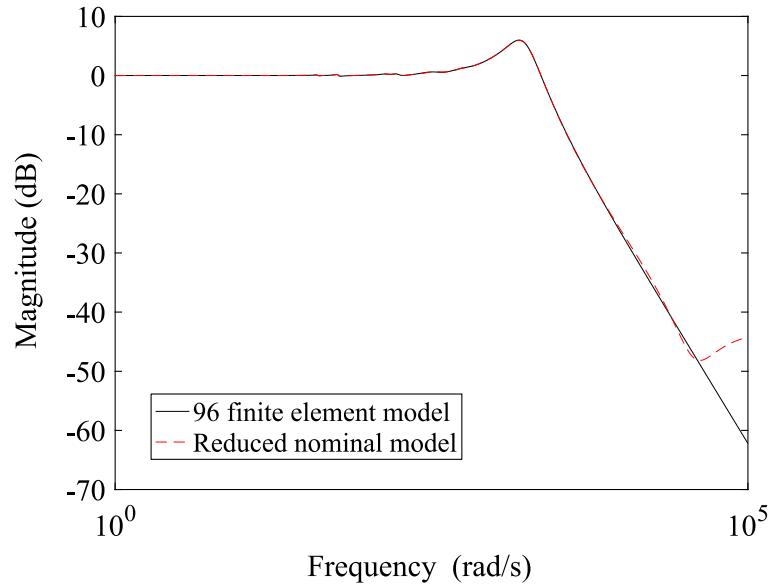
#### 4.1.2 Subsystem coupling and uncertainty refinement for coupled system

Up until now the author only considered modeling uncertainty for a single subsystem. One can stress that this series of successively better subsystems models are essentially local since one didn't take into the account that the overall system is comprised of a series of coupled individual instances. To address this issue, let us propose a procedure for modeling uncertainty weights for a single subsystem that takes into the account the neighboring subsystems coupling. Information about the coupled neighboring subsystems is used to further reduce the previously obtained uncertainty weights. A reduction of required uncertainty weights means a reduction in conservatism of an uncertain system. Since the author is analyzing passive and dissipative coupled system author is assuming that a portion of individual subsystem uncertainty is damped by the neighboring subsystems and may be discarded. It needs to be kept in mind that this reduction of uncertainty might not be the case for spatially distributed system in general. Reduction of uncertainty using this presented approach may even vary for different passive and dissipative systems depending on the system structure and properties. The procedure for obtaining unstructured uncertainty for a single subsystem can be used with a minor extension in a following manner.



**Figure 4.3:** Input multiplicative uncertainty model: a) subsystem  $G_{sys}$  coupled with reduced model of neighboring subsystems  $G_{red}$ , b) representation as a general framework for an uncertain feedback system

First, one needs to take into the account the neighboring systems. The neighboring systems are series of a final number of previously obtained reduced nominal models. If one further reduces this series, one can obtain a reduced model  $G_{red}$  that represents the neighboring systems. For reduction of a series one can use balanced truncation with same criteria used for obtaining the nominal (single subsystem) model. Finally, one complete the modeling by connecting  $G_{red}$  into feedback loop with the uncertain individual subsystem  $G_{sys}$  as shown in Figure 4.3a. The resulting model can be viewed as a general framework for robust control, where an uncertain feedback system (i.e.  $G_{sys}$  with  $\Delta$  in feedback) has  $G_{red}$  as a controller on a system as shown in Figure 4.3b. With this model one can study the effect of neighboring systems on unstructured uncertainty model refinement by essentially changing the number of surrounding systems in  $G_{red}$ . Author stresses that the number of neighboring systems needed for this certainly depends on the problem at hand. For this case study one can chose that the number of surrounding subsystems change, i.e. connect a different number of reduced nominal models in a feedback loop. For each number of neighboring subsystems, that is 1,3, 5 or 10 reduced nominal models, one can recompute input multiplicative frequency weights  $W_1$  and  $W_2$  in a similar fashion as it was done for a single subsystem. It is important to note that  $G_{red}$  for each number of neighboring systems becomes bigger in size but is nevertheless kept reasonably low.  $G_{red}$  has 19, 25, 29 and 40 states for 1, 3, 5 and 10 neighboring systems, respectively. With the complete mathematical model, let us present results as shown in next chapter.



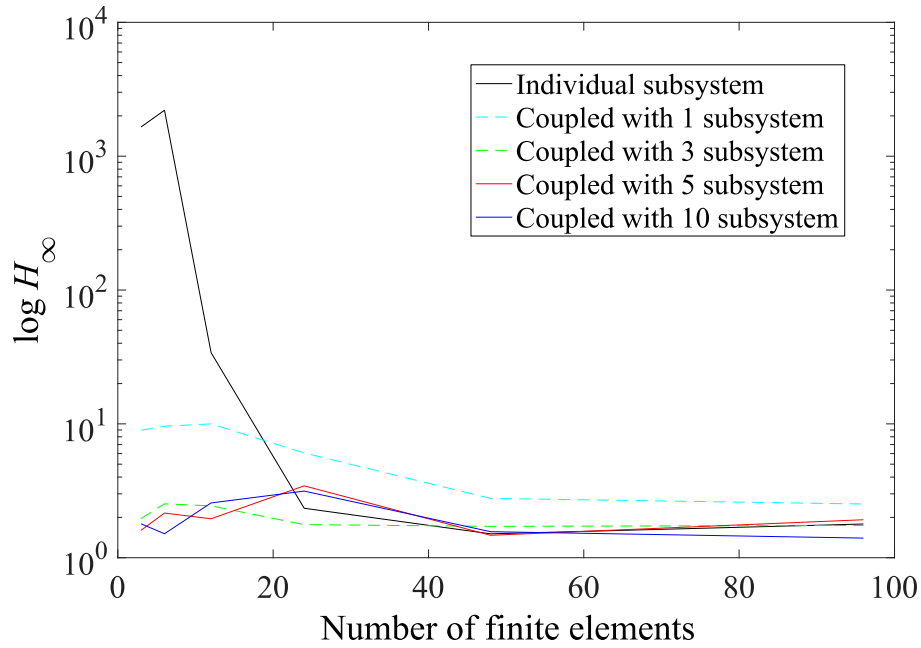
**Figure 4.4:** Frequency response of the most accurate model from 96 FE and a nominal reduced model

### 4.1.3 Discussion of results for the motivational example

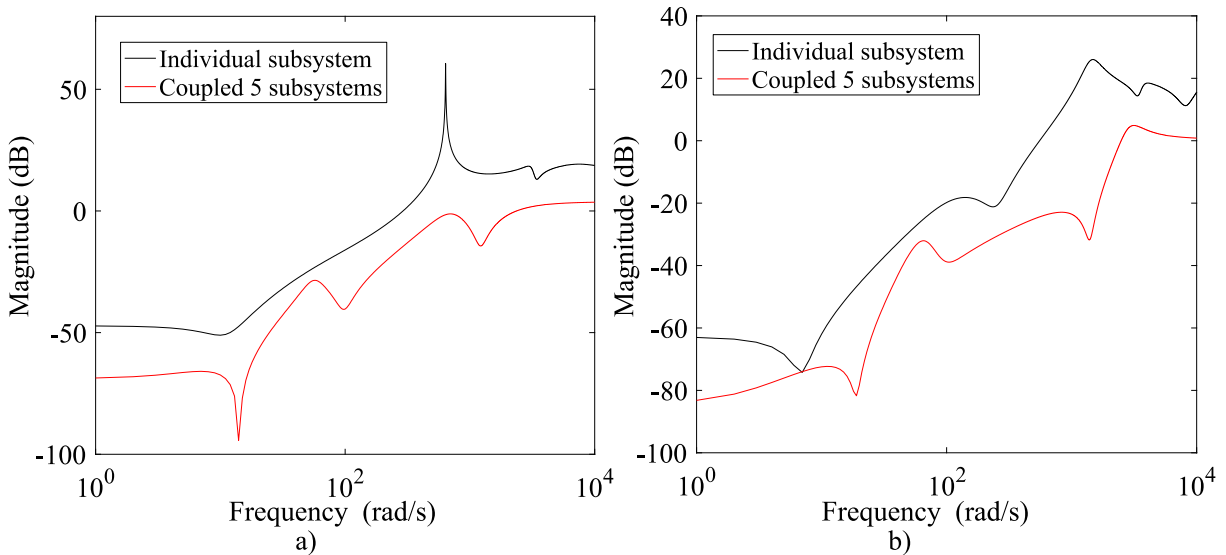
To keep the size of a nominal model small, let us reduce the most accurate system (i.e. the one with 96 FE). The response of a 96 FE model and that of a reduced model is shown in Figure 4.4 where it can be seen that the reduced model accurately describes the response of the full model.

First, let us study frequency weights obtained for single subsystem. A comparison of  $H_\infty$  norms of frequency weights for each discretization level is shown in Figure 4.5. As expected, the frequency weights norm decreases with finer discretization. This provides us quite straightforward criteria for deciding which discretization should be used. For example, a criterion could be such that the frequency weight norm is less than some prescribed value.

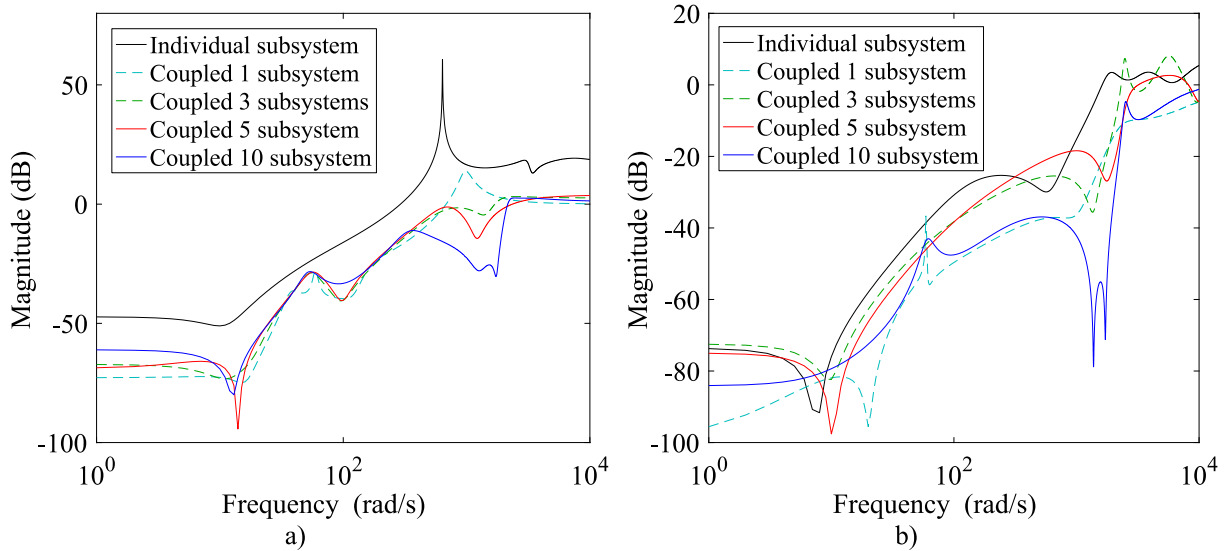
Next, let us compare  $H_\infty$  norms for coupled subsystems when one adds 1, 3, 5 and 10 neighboring subsystems. A comparison of  $H_\infty$  norms of frequency weights for both single and coupled subsystems for different levels of discretization is shown in Figure 4.5 as well. It is evident that new  $H_\infty$  norms obtained for coupled dynamical systems are decreasing. Most prominent decrease can be seen for low discretization levels. When analyzing obtained frequency weights for individual subsystem and coupled subsystem it is seen that the newly obtained frequency weights for coupled subsystem have lower magnitude of the frequency response in wide frequency ranges as shown in Figure 4.6. In other words, the frequency weights of individual uncertainty models are “scaled” such that the uncertainty information that does not affect the overall system dynamics is discarded.



**Figure 4.5:** Comparison of  $H_\infty$  norms of frequency weights for a single subsystem and a different number of coupled subsystems for each discretization level



**Figure 4.6:** Comparison of frequency response of frequency weights for a single subsystem and a coupled subsystem for: a) 3 FE model, b) 12 FE model



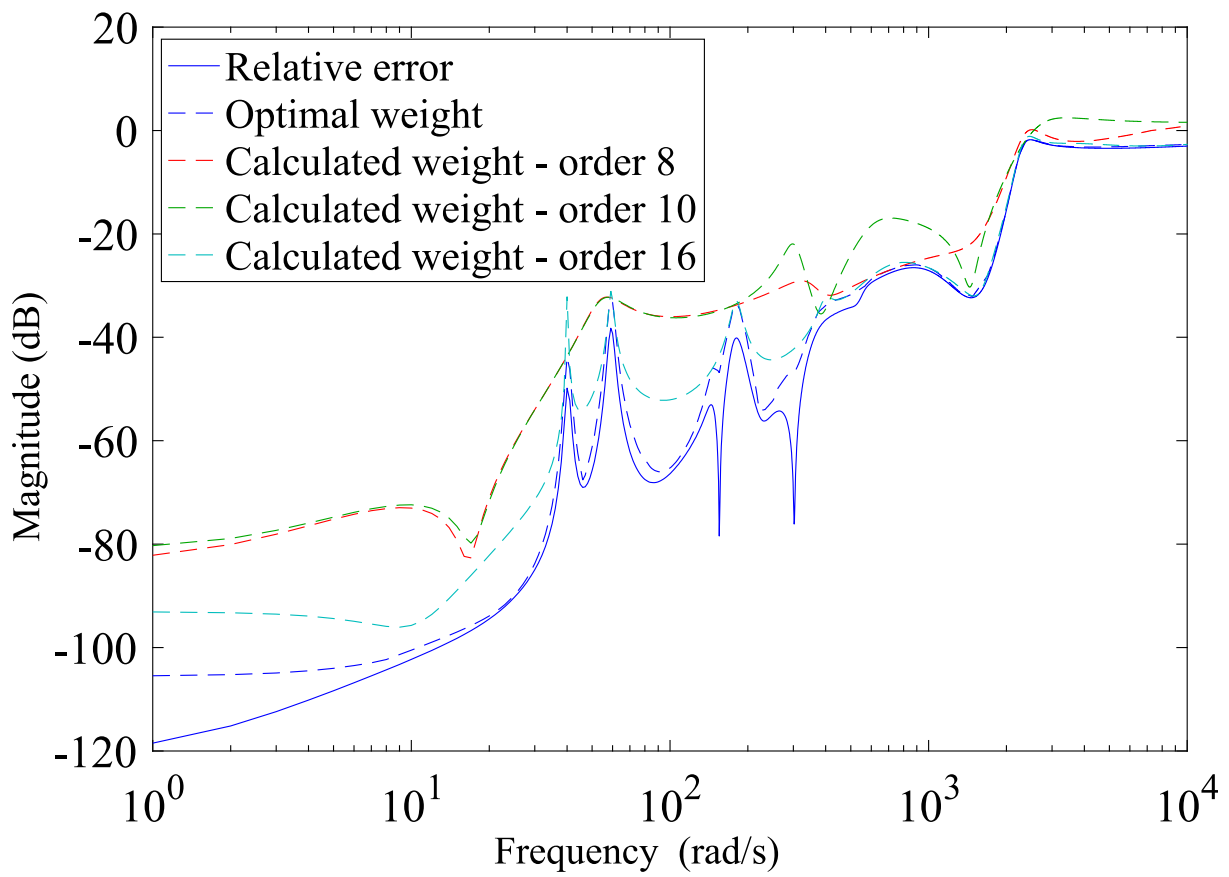
**Figure 4.7:** Comparison of frequency response of frequency weights for a single subsystem and a different coupled subsystem for: a) 3 FE model, b) 24 FE model

When changing the number of surrounding systems (i.e. by adding more neighboring subsystems), uncertainty model can be further reduced, but this time only in a specific spectrum, as seen in Figure 4.7. For this example, higher frequency dynamics at around  $10^3$  rad/s is discarded from uncertainty model when adding more neighboring subsystems.

Numeric algorithm used for obtaining input multiplicative uncertainty weights calculates optimal weights on a frequency grid. It is interesting to investigate how well different order of weights covers the relative error between the nominal system and the LTI model for some discretization as it is shown in Figure 4.8. An increase in order of weights from 8 to 10 and then to 16 results in higher order weights more closely matching the response of relative error. This increase in weight order comes at cost of increase in size of the uncertain model and an increase in numeric calculation times.

An interesting observation is made when applying  $\nu$ -gap analysis by Vinnicombe [12]. For the problem at hand a reduced series of neighboring systems can be viewed as a controller on an uncertain system as shown in Figure 4.4b. It is thus interesting to observe upper bounds on the Vinnicombe gap and distances between each controller model when let us increase the number of neighboring systems. That is, one can measure upper bounds on  $\nu$  and  $\nu$ -gap between one system connected as a controller and 3 systems connected as a controller on an uncertain system, and so on. It was shown that even controller models obtained by adding 1 subsystem and 2 subsystems in feedback are far apart in the sense that  $\nu$ -gap is approximately 0.96. Between each other combination of controllers  $\nu$  and  $\nu$ -gap are practically equal to 1 which translates into systems being far apart [131]. Also, to exclude possible errors in  $\nu$  and  $\nu$ -gap calculation instead of using reduced controller, analysis was carried out on a





**Figure 4.8:** Comparison of frequency response of frequency weights for coupled subsystem and a different order of weights for 12 FE

full model as well and yielded identical results. It would be interesting to further investigate these results as well.

The presented are results on the modeling of unstructured uncertainty for spatially distributed (coupled) vibration systems. Finite element mesh refining technique is used to obtain a series of successively finer spatial discretizations of an infinite dimensional vibrational mechanical system. This refining provides enough information for constructing a series of successively better uncertain linear time-invariant models. The resulting uncertain linear time-invariant model of a single subsystem can be further refined for a special class of passive and dissipative vibration systems by taking into account that it is coupled with a small number of neighboring subsystems that are passive and dissipative as well. By increasing the number of neighboring systems, it was shown that the magnitude in frequency response of required uncertainty on the single subsystem is decreasing. This reduces conservatism of the uncertainty model of individual subsystem which improves performance when dealing with robust control of passive and dissipative coupled dynamical vibration systems.

## 4.2 Structure Preserving Uncertainty Modelling

To model uncertainties for spatially discretized and reduced order spatially distributed dissipative dynamical systems that are suitable for (distributed) robust controller synthesis, an coupled system consisting of many subsystems with as low-order and as least conservative uncertainties as possible, arising from both discretization and MOR, has to be obtained.

As seen from Equation (3.1) and the definition of  $\Delta_j$  only being bounded by the  $\mathcal{H}_\infty$ -norm, this type of uncertainty is often called unstructured uncertainty. With this in mind, the *uncertainty modelling* essentially boils down to choosing the appropriate weights that essentially *adjust* (or *scale*) the amount of required uncertainty at each frequency. In practice, the weights are usually some combination of low-pass, band-pass and high-pass filters [60, 128]. Indeed, as it is recognized in the literature, modelling of appropriate weighting filters (in general) is perhaps one of the most important jobs of a system engineer [129]: “. . . design of a robust and high-performance control system relies heavily on the choice of weights used in the design. . .”. Modelling the frequency weights, allows the control designer to capture the system uncertainties and to determine over what frequency ranges the performance is desired. Weights can be viewed as a way to fine tune the control design to achieve a desired level of performance and robustness without having to explicitly consider the system stability [132].

The author stresses that the uncertainty conservatism reduction method presented so

far is essentially *local* [133]. As it will be shown in the sequel, uncertainties modelled with structure preservation in mind can be further improved, thus, the subsystem uncertainty conservatism can be further reduced. Due to the mutual interconnections of the subsystems, and the fact that the subsystems and/or interconnections are dissipative, part of the uncertainty (especially in the high frequency range) can be discarded. The theoretical part regarding the definition of dissipative systems can be found in [60], while the simple explanation can be given in terms of energy loss throughout the system. The idea is to reduce the uncertainty conservatism of each subsystem, thus reducing the conservatism of the overall coupled system as well, by studying energy losses throughout the system. However, in the meantime, it is also important to keep the order of the coupled system as low as possible. While doing both of these things, the following mandatory things need to be achieved: (i) the reduced order discretized coupled subsystem needs to be robustly stable; (ii) the order of the coupled system should be kept as low as possible and; (iii) robust performance of the reduced order discretized coupled system needs to be as close as possible to the performance of the reference coupled system. To achieve the points stated previously, author proposes a novel design procedure that is explained in the next section.

### 4.2.1 Design procedure

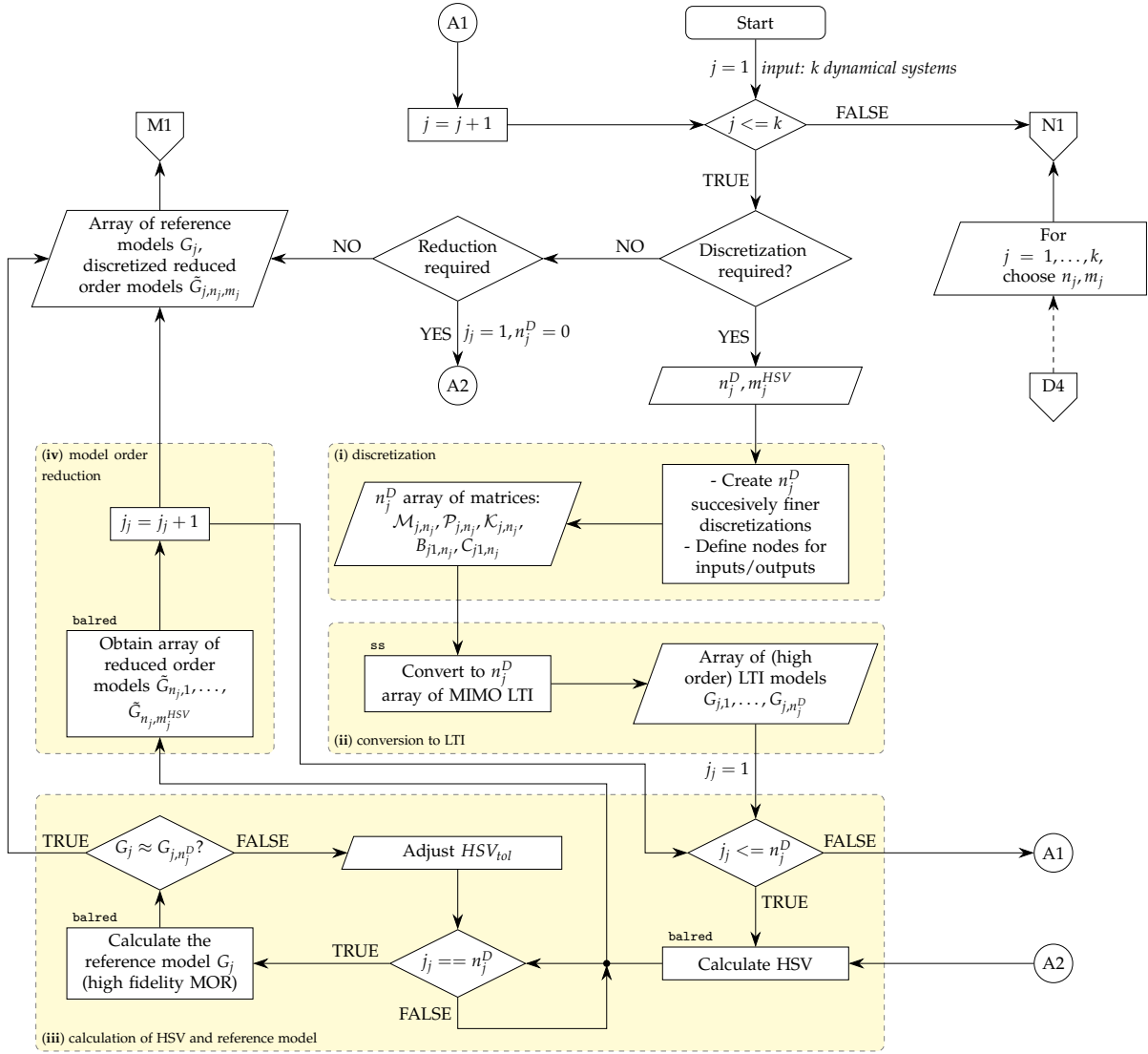
The newly developed design procedure will be written out as *ready to implement in MATLAB philosophy*, so wherever appropriate, useful MATLAB commands and routines will be highlighted. For the complete usage and compatibility, latest (as of date of article publishing) version of MATLAB, Control system toolbox™ and Robust control toolbox™ should be used. In the IQC robustness analysis part of the procedure, commands from the IQCLab Toolbox (V3.0) [68] are also mentioned.

To calculate the absolute error and additive uncertainty, an array of models is created. Reference models  $G_j$  are obtained from the finest discretization. The discretized reduced order models are distinguished by the discretization using index  $n_j = 1, \dots, n_j^D$  and the order (states kept) using index  $m_j = 1, \dots, m_j^{HSV}$ . For simplicity one will often omit these indexes and just write  $\tilde{G}_j := \tilde{G}_{j,n_j,m_j}$ . This process of obtaining an array of reference and discretized reduced order models is shown in Figure 4.9 and represented with steps (i) to (iv) therein.

Let us continue by choosing appropriate discretizations ( $n_j$ ) and orders ( $m_j$ ) of reduced order models for each subsystem. After that, the idea is to scale both the reference model and the discretized reduced order model to obtain a scaled version of the absolute error denoted as  $G_{j,e} - \tilde{G}_{j,e}$  for each subsystem. Author stresses that these scalings can be directly calculated from the structure of the system and the fact that the subsystems are dissipating energy will be potentially beneficial—one can call these

scalings input-output transfer functions (IOTFs) and details on how to obtain these are detailed later in Sections 5.2.2 and 5.2.3. This process is shown in Figure 4.10 and denoted as a step (v). Besides the novelty of the overall procedure—or the approach to the discretization and model order reduction error uncertainty modelling—this step (i.e., (v)) also represents a first major novelty of the procedure. The fact that using only the existing structure of the system it is possible to reduce the uncertainty conservatism is something not yet seen.

With scalings (IOTFs) calculated, a scaled version of the absolute error,  $G_{j,e} - \tilde{G}_{j,e}$ , can be calculated for each subsystem. Due to the dissipative nature of the coupled system, some of the energy is lost when coming to and from each subsystem, such that if one compared the gain of  $G_{j,e} - \tilde{G}_{j,e}$  in comparison to the unscaled version of it, i.e.,  $G_j - \tilde{G}_j$ , a significantly lower gain over a wide frequency range can be observed. As stated before, the main idea of additive uncertainty modelling is to replace these errors with the appropriate frequency weights—thus let us introduce weights  $W_{j,1e}$  and  $W_{j,1}$  to capture the dynamic behaviour of the aforementioned errors. Steps for obtaining the uncertainty weights, a scaled version and an unscaled version, for each subsystem, are shown in Figure 4.12 denoted with (vi) and (vii), respectively. The second main novel part is being two fold. First, by observing the dynamic behaviour, of the scaled absolute error particularly, and to a lesser degree of the unscaled absolute error as well—there usually are no high peaks in gain. As such, it makes sense to use the weights of low order to capture such a behaviour. This can efficiently be obtained using low order logarithmic-Chebyshev magnitude filter design—the details on this particular choice of filter are elaborated in Section 5.2.1. The second part of the second major novelty comes from the observation, that in many cases, one can completely reduce the order of the filter—by simply choosing a filter of static (constant) gain—thus no additional states are introduced in the model at increase in conservatism.



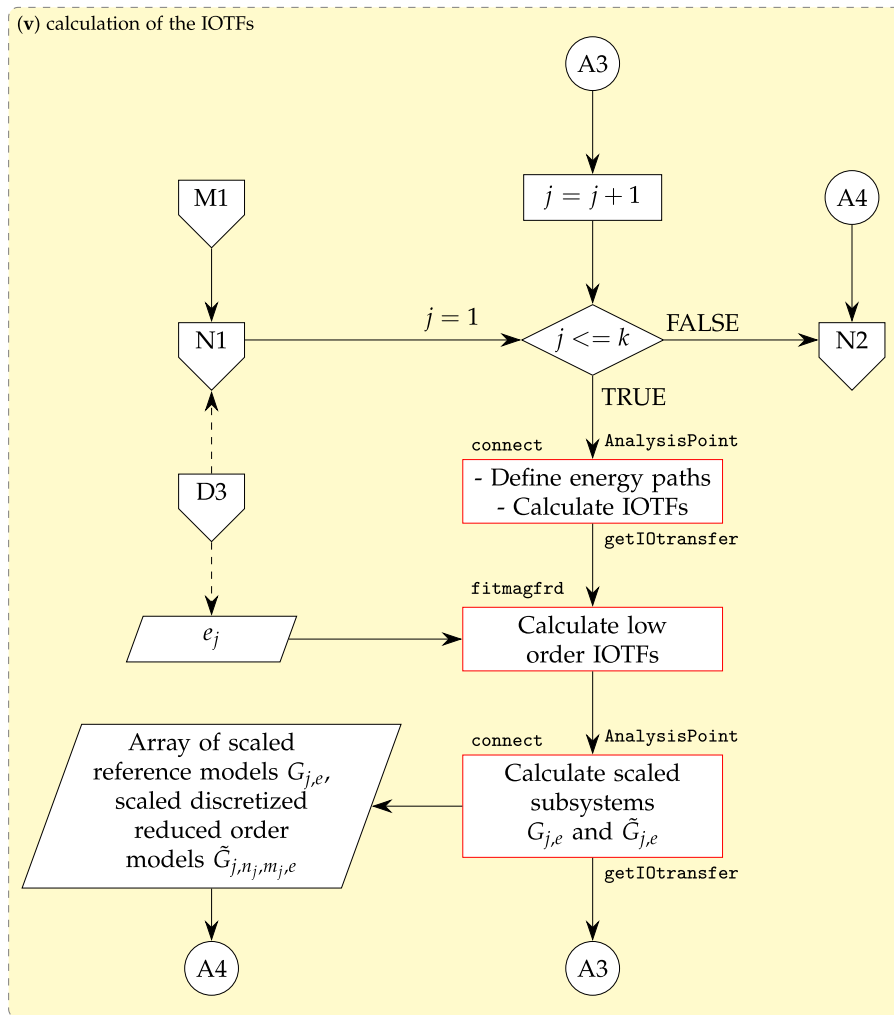
**Figure 4.9:** A preprocessing step for obtaining and storing of arrays of LTI models: reference models ( $G_j$ ) and discretized reduced order models ( $\tilde{G}_{j,n_j,m_j}$ ). For each discretized system,  $n_j = 1, \dots, n_j^D$  discretized LTI models are made. For discretized models, a reference model is created by choosing the best available discretization (i.e.,  $n_j = n_j^D$ ) that is reduced using high fidelity balanced truncation method—a step analogous to obtaining a minimal realization. For each reduced order model, Hankel singular values are calculated and only first  $m_j = 1, \dots, m_j^{HSV}$  models are stored. It should be noted that the discretization step (i) can be carried out using readily available FEA/FEM or meshing software.

Since this fact (of not increasing the order of the uncertain system) comes with practical usability, from now on, the focus is put mostly on these weights, i.e.,  $W_{j,1s}$ , to a lesser degree on the scaled (refined) weights  $W_{j,1e}$  and only keep in mind the unscaled (unrefined) weights  $W_{j,1}$  that are overly conservative for a comparison, to demonstrate the results. All the dynamic behaviour will be captured only by weights  $W_{j,1s}$  (or  $W_{j,1e}$  or  $W_{j,1}$ ), while the second weight will be kept to be identity. A more elaborate descrip-

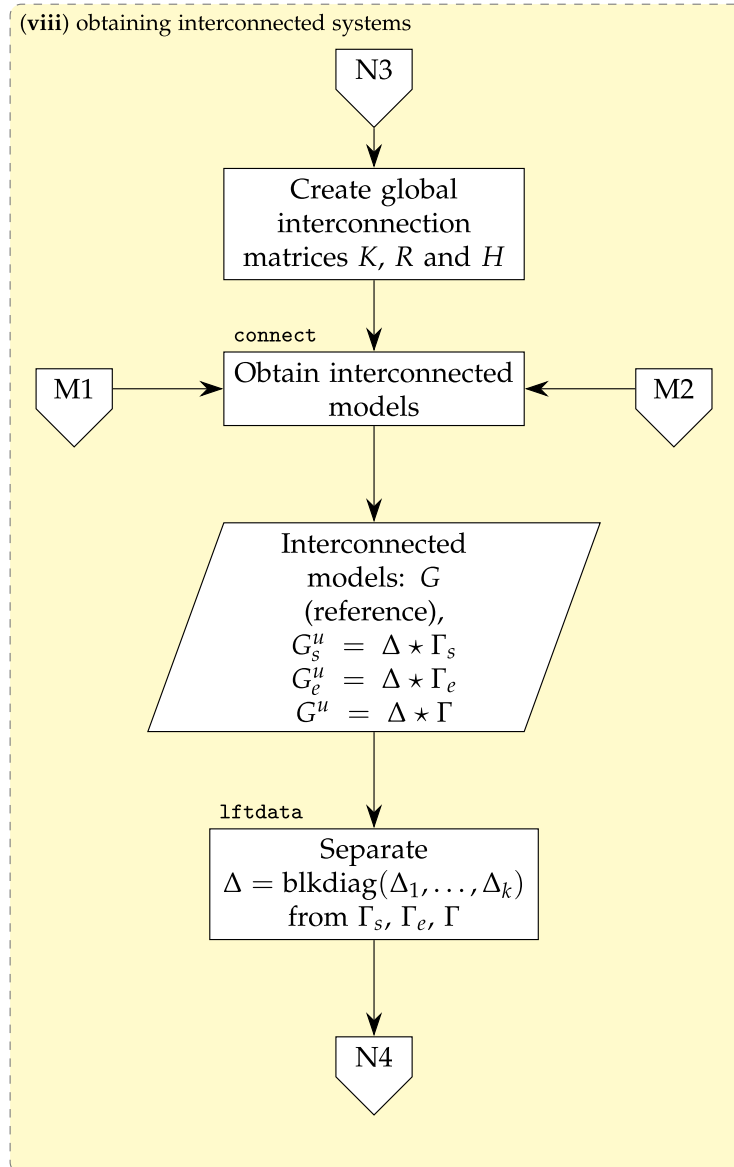
tion on when one might use  $W_{j,2}$  is given in Section 5.2.4. Moreover,  $\Delta_j$  per subsystem always stays the same. With that being said, an array of uncertain subsystems can be created.

With the arrays of reference models and uncertain systems ready, as shown in Figure 4.11, denoted as a process (vii), one can obtain coupled models. Here it should be noted that interconnection matrices  $R$ ,  $K$  and  $H$  can be reused to calculate reference coupled system  $G$  (used only for comparison and verification) and uncertain coupled system(s). Again, let us keep the attention at the uncertain coupled system  $G_{j,s}^u$ , while the other uncertain system  $G_{j,e}^u$  can have practical usage, the uncertain system  $G_j^u$  is only used to demonstrate the effect of uncertainty conservatism reduction (often called uncertainty refinement).

The last step of the process is to carry out robustness analysis. In this paper, the focus is on the robustness analysis using integral quadratic constraints (IQCs). Without going into details, the key point can be explained as follows. For a given coupled uncertain system (previously converted to a LFT form)—again, focus is on  $G_{j,s}^u$ —find a feasible solution to prove that the given uncertain system is robustly stable. After that, check the best achievable  $\gamma$  and compare it to the induced  $\mathcal{L}_2$ -gain of a reference coupled system. This part of the procedure is shown in Figure 4.14 and denoted as (x).

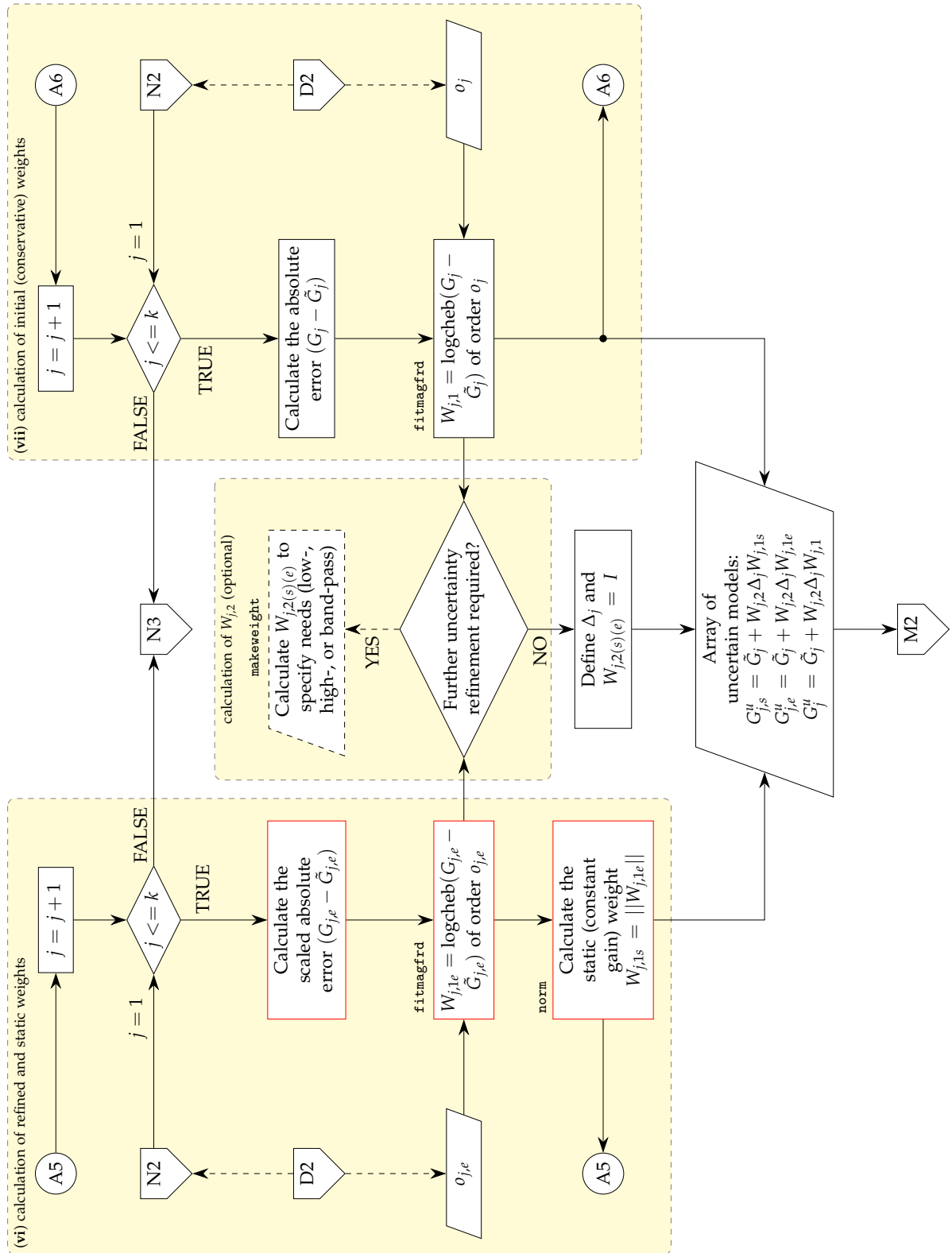


**Figure 4.10:** Structure of the system is used to obtain the insight into how energy is being dissipated (damped) inside the system. Interconnection matrices are used to create coupled reference system and coupled uncertain systems. Complex interconnections (or surroundings, environment) around each subsystem are essentially acting as scaling filters at input and output of each subsystem (input output transfer functions-IOTFs). These scaling filters (IOTFs) are reduced and used to calculate a low order scaled versions of both the reference model and discretized reduced order model.



**Figure 4.11:** Structure of the system is used to obtain the insight into how energy is being dissipated (damped) inside the system. Interconnection matrices are used to create coupled reference system and coupled uncertain systems. Once the interconnection matrices  $R, K$  and  $H$  have been created, and a reference coupled system is obtained (i.e.,  $G$ )-uncertain coupled system(s) can also be obtained using the same interconnection matrices and replacing the appropriate reference subsystems with the relevant uncertain subsystems (i.e., replace  $G_j$  with  $G_{j,s}^u$ ,  $G_{j,e}^u$  or  $G_j^u$ ).



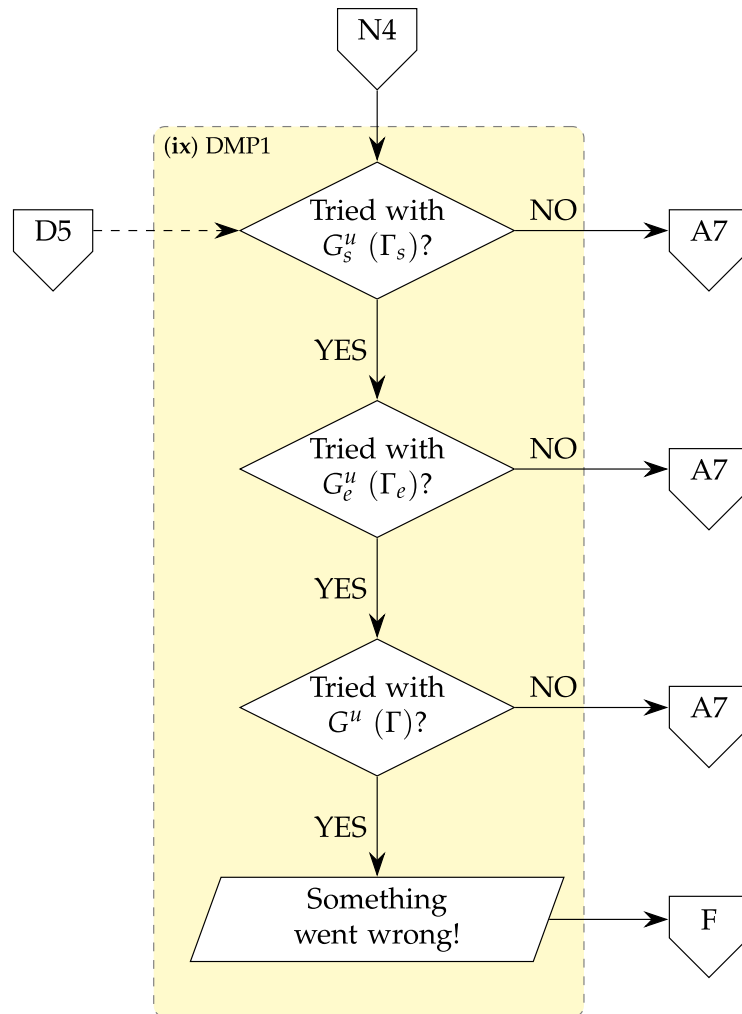


**Figure 4.12:** Process of uncertainty modelling and uncertainty refinement through the creation of weights  $W_{j,1}$  and  $W_{j,1e}$ . Calculating the absolute error and obtaining weights for each subsystem, without taking into account the (dissipative) surroundings of the subsystem yields conservative results (i.e.,  $W_{j,1}$ , a process denoted as (vii)). Calculating the absolute errors for each subsystem that is scaled by its surroundings and turning it into refined weights  $W_{j,1e}$ , results in lower magnitude

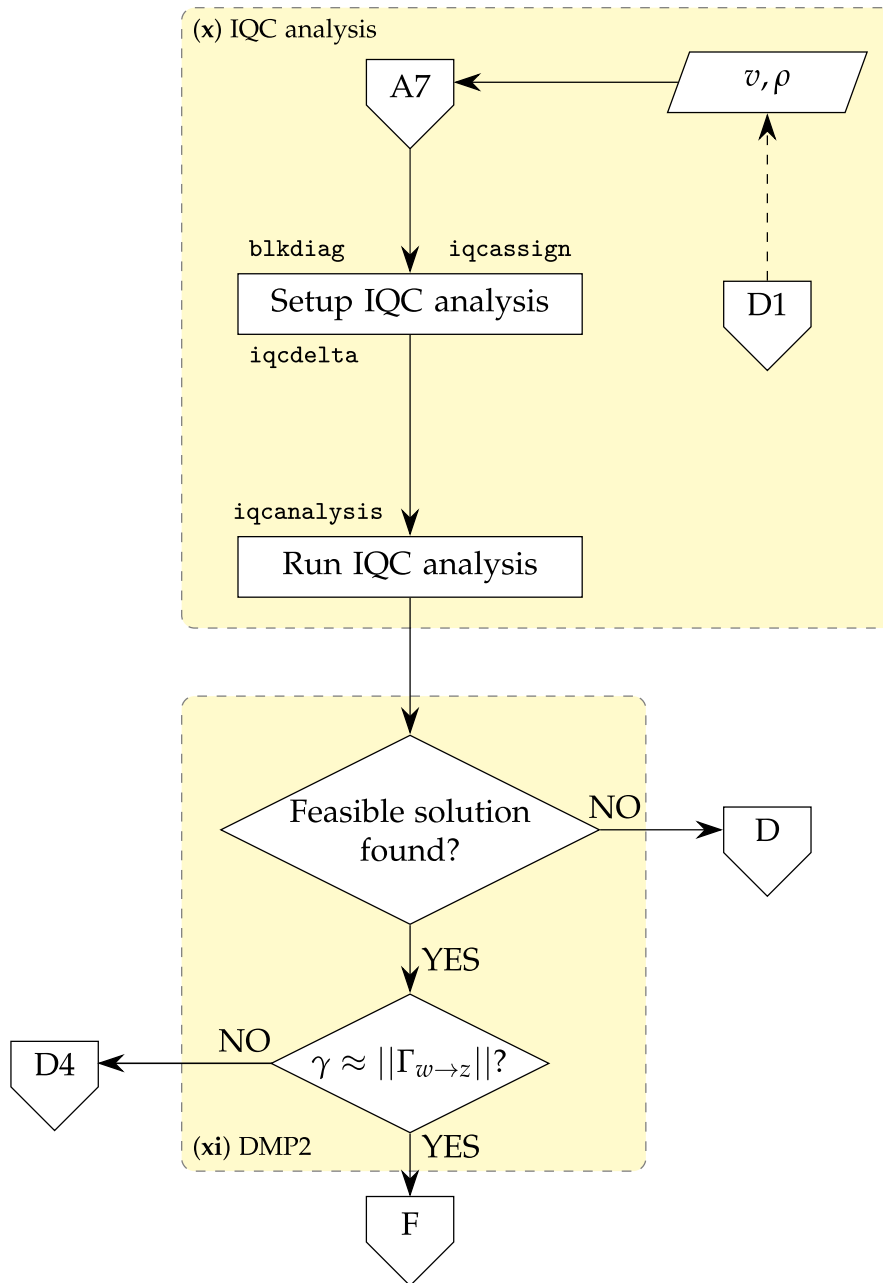
in frequency response in wide frequency range—hence less conservative uncertainty model was achieved. The scaled weights can often be further simplified by simply replacing it by its peak response—this offers a flexible and useful trade-of—reduce the order of the weights (i.e., no additional states are introduced with weights of static (constant) gain response) at introduction of some conservatism, or vice versa. The steps needed to obtain the weights of static gain  $W_{j,1s}$ , are denoted as (vi).

It should be noted that since the proposed procedure deals with the uncertainty modelling (or modelling in general), many parts of the process require manual operation. For example, the initial guess of the required discretization and/or orders of reduced order models (i.e.,  $n_j$  and  $m_j$ , respectively) a feasible solution might be found immediately and the performance criteria met easily. On the other hand, it might be a complete miss. Here, however, a main paradigm when choosing discretization and orders is to model systems close to the external inputs and outputs with relatively fine discretization and relatively high orders; and the one *far* away with coarse mesh and low orders. This also intuitively makes sense to do, since the dynamics of the distant subsystems is not affected by the external inputs or does not influence the measured outputs. With that being said, one can complete the design procedure by proposing a useful decision making process (DMP), as shown in Figure 4.16, that can be used to efficiently find an uncertain coupled system for which a robustness will be guaranteed. Author stresses that carrying out robustness analysis using  $\mu$ -tools instead of IQC analysis is as easy as reformulating questions in the decision making process of the IQCs analysis (see (xi) DMP2 in Figure 4.14)—where “Feasible solution found?” becomes “Robust stability margin greater than 1?” and “ $\gamma \approx ||\Gamma_{w \rightarrow z}||$ ” becomes “Worst case gain close to  $||\Gamma_{w \rightarrow z}||$ ?”.

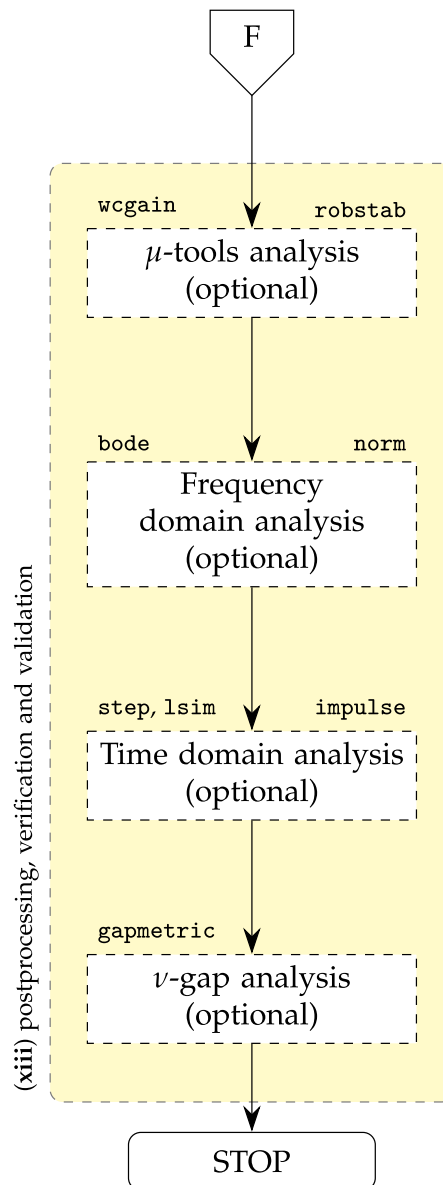
One more note will be given before proceeding to the numerical example. While in the proposed design procedure one exclusively used either static  $W_{j,1s}$  or refined  $W_{j,1e}$  weights to obtain the uncertain coupled system—an important advantage of the structure preserving approach is that one can combine static and refined weights per each subsystem (i.e., some subsystems’ uncertainty is modelled using static weights and for some other using refined weights). This gives the opportunity to further fine tune a trade-of between reduction in the uncertainty conservatism and the order of the coupled uncertain system. It is left to the interested readers as a mental exercise to see how this can be utilised properly for a specific problem—because the modelling of each new complex coupled dynamical system is a problem for itself.



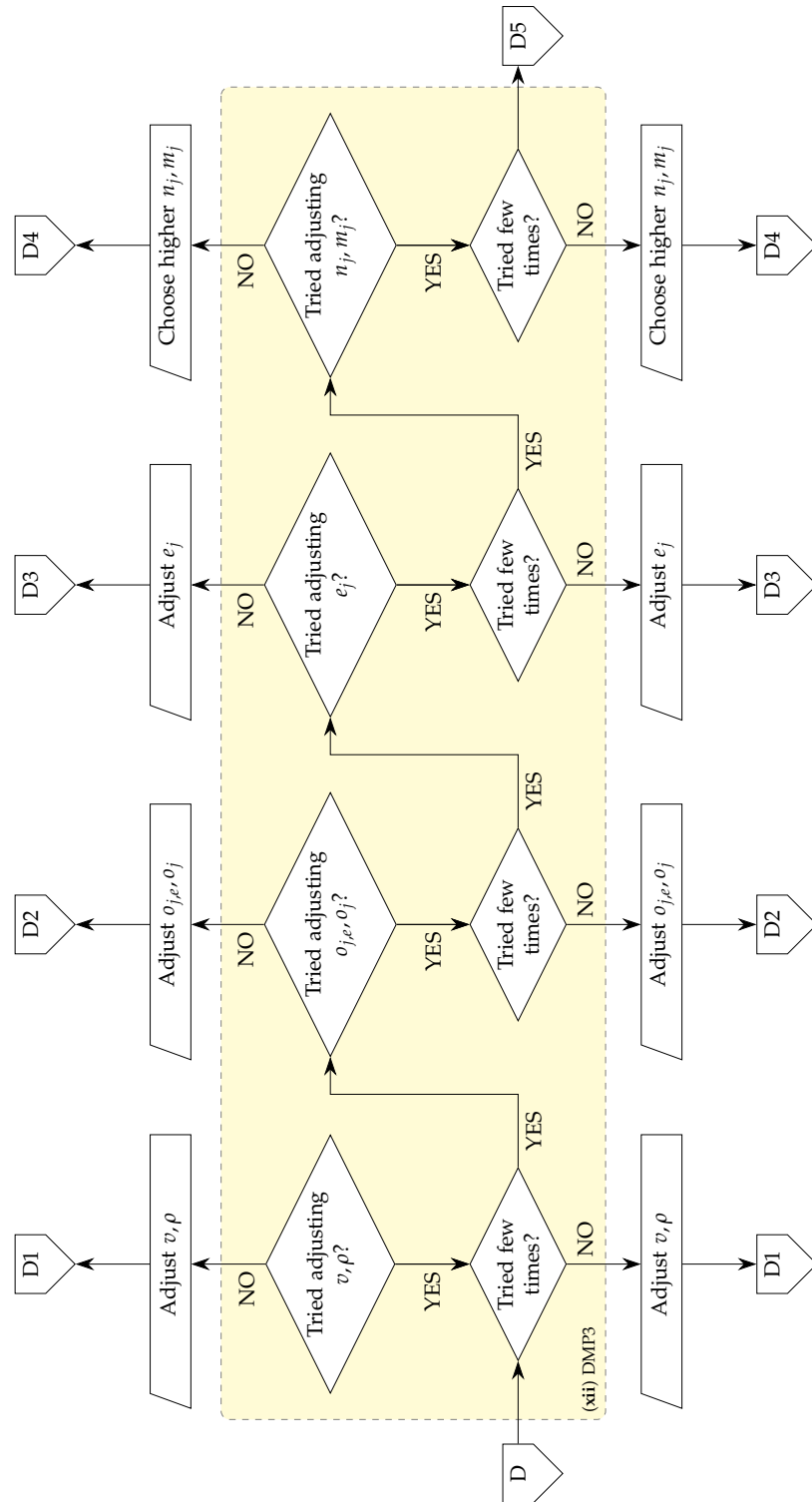
**Figure 4.13:** Carry out IQC analysis and optional postprocessing (if the IQC analysis was successful) for a given nominal parts of uncertain system(s)  $\Gamma_s$  ( $\Gamma_e$ ,  $\Gamma$ ) and check if all the robustness criteria are satisfied. Decision making process 1 (DMP1) — if a feasible solution cannot be found and/or performance criterion cannot be met with a  $\Gamma_s$ , try  $\Gamma_e$  or  $\Gamma$ .



**Figure 4.14:** Carry out IQC analysis and optional postprocessing (if the IQC analysis was successful) for a given nominal parts of uncertain system(s)  $\Gamma_s$  ( $\Gamma_e, \Gamma$ ) and check if all the robustness criteria are satisfied. IQC analysis process and decision making process 2 (DMP2)-if there is no feasible solution found to the IQC analysis appropriate changes have to be made; either to the IQC analysis parameters or the uncertain system (see Figure 4.16 for further details), if there was feasible solution found, but the specified performance criterion has not been met, then the discretizations and/or orders of the reduced order models are too low and have to be adjusted.



**Figure 4.15:** Carry out IQC analysis and optional postprocessing (if the IQC analysis was successful) for a given nominal parts of uncertain system(s)  $\Gamma_s$  ( $\Gamma_e, \Gamma$ ) and check if all the robustness criteria are satisfied. Continue to optional post-processing—i.e., carry out robustness analysis using  $\mu$ -tools instead of IQCs, run  $\nu$ -gap analysis or perform frequency and/or time domain simulations to confirm the desired behaviour of the obtained uncertain coupled system.



**Figure 4.16:** Decision making process 3 (DMP3) — The suggested order of adjustments that have to be made to the IQC analysis parameters and to the uncertain coupled system to satisfy a given robustness criteria. First decision (D1) reflects the changes are being made to the IQC parameters (i.e.,  $v$  and  $\rho$ ). Other decisions reflect the changes are being made to the uncertain coupled system. Changes are being made to the orders of uncertainty weights first (D2), then to the orders of reduced order IOTFs (D3) and after that to the level of discretizations of subsystems (i.e.,  $n_j$ ) and/or to the orders of the model order reduced subsystems (i.e.,  $m_j$ )—collectively the fourth decision (D4).

## CHAPTER 5

# Numerical Experiments and Discussion

### CHAPTER CONTENTS

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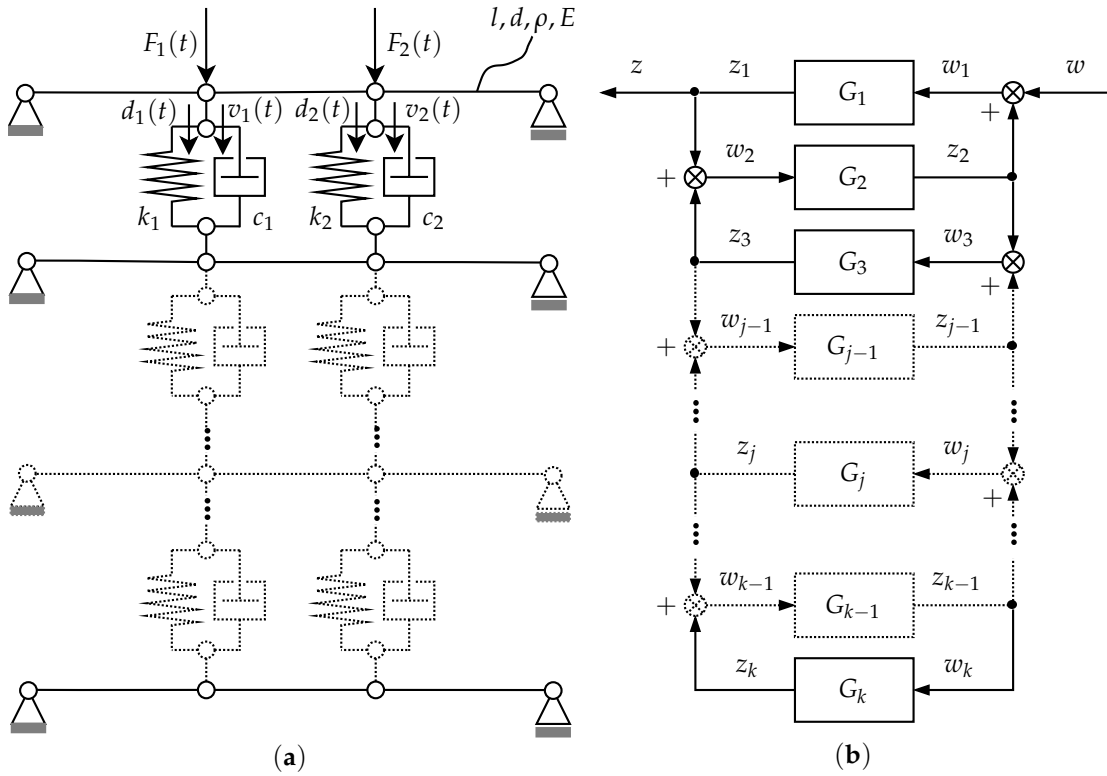
## 5.1 Series of simply supported Euler beams coupled by springs and dampers: A structure preserving approach

All the code needed to replicate the results shown in the sequel is available in GitHub [134] and archived in Zenodo [135].

Let us now consider the application of the proposed procedure on the practical example—a series of simply supported Euler beams mutually interconnected by springs and dampers. Such a system is shown in Figure 5.1a. All beams of equal length  $l$ , divided into three ( $n_s = 3$ ) equal length segments, the same circular cross section with diameter of  $d$ , the same mass density  $\rho$  and the same Young’s modulus of elasticity  $E$ . On the uppermost beam, at one third and two thirds of its length,

two dynamical vertical forces  $F_1(t)$  and  $F_2(t)$  are applied. At the same locations, displacements of the beam  $d_1(t)$  and  $d_2(t)$ , and velocities  $v_1(t)$  and  $v_2(t)$ , respectively, are measured. On those same locations, spring-damper pairs are connected to the next successive beam. Internal inputs to the beams are forces acting from spring and dampers to the nodes at the beams, while the internal outputs are displacements and velocities of nodes at beams. Material properties of the beams, as well as stiffnesses of the springs and viscous damping coefficient of the dampers, are shown in Table 5.1 for three different test cases. The zero values for  $c_1, c_2$  or  $k_1, k_2$ , mean completely disconnected dampers or springs, respectively, thus obtaining systems with less internal outputs. The beams are discretized using 2D Euler–Bernoulli beam finite elements. Ten ( $n_d = 10$ ) successively finer discretizations are made, such that each of discretized models have a number of 2D beam finite elements (FE) defined as  $n_{FE} \in \{3n_s n_d \mid n_d = 1, \dots, 10\}$ . FE nodes are equally distributed along the length of the beam. After constructing the global mass and stiffness matrices, a proportional Rayleigh damping matrix is calculated such that the modal damping for the first 8 beam vibration modes ratio is approximately  $\zeta$ .





**Figure 5.1:** A series of simply supported Euler beams mutually interconnected by springs and dampers: **(a)** Mechanical schematic of an interconnected system. **(b)** Block diagram representation of the system, where the systems with odd indices, i.e.,  $G_1, G_3, G_j, \dots, G_k$  are dynamical systems representing LTI models of discretized beams, while the systems with even indices, i.e.,  $G_2, G_4, G_{j-1}, \dots, G_{k-1}$  are the systems of static gain representing springs and dampers between each beam.

**Table 5.1**  
Material properties of the beams, springs and dampers.

Case #	$l$ m	$d$ $m^2$	$\rho$ $kg/m^3$	$E$ GPa	$c_1$ $N \cdot s/m$	$k_1$ N/m	$c_2$ $N \cdot s/m$	$k_2$ N/m	$\zeta$ -
1	2	0.018	7800	$210 \times 10^9$	$10^{-4}$	0	$10^{-6}$	0	0.08
2	2	0.01	7800	$210 \times 10^9$	0	$7 \times 10^0$	0	$3 \times 10^1$	0.08
3	1	0.01	7800	$210 \times 10^9$	$10^{-2}$	$2 \times 10^2$	$10^{-1}$	$10^1$	0.05

All the discretized systems are converted to LTI state-space systems using Equations (2.1) to (2.3). Using Equations (2.3) to (2.8), interconnection matrices are obtained, with input to each subsystem being  $w_j(t) = [F_{1,j}(t), F_{2,j}(t)]^T$ , output of the each subsystem being  $z_j(t) = [v_{1,j}(t), v_{2,j}(t), d_{1,j}(t), d_{2,j}(t)]^T$ ,  $F_j(t)$ , while  $w(t) = [F_1(t), F_2(t)]^T$  and  $z(t) = [v_1(t), v_2(t), d_1(t), d_2(t)]^T$ , being performance (exter-

nal) inputs and outputs, respectively. Spring-damper pairs between each beam can be represented as systems of static gain. This interconnected system can be represented as shown in Figure 5.1b. From coarsest to finest mesh, LTI state space models have 36, 72, 108, 144, 180, 216, 252, 288, 324 to 360 states, respectively. The HSV are calculated (but also stored for later usage) and high fidelity MOR is carried out such that all the states with HSV less than  $10^{-12}$  are truncated, resulting in systems that has 24, 40, 46, 50, 52, 54, 57, 58, 60 and 62 states. Comparing the systems to the original ones, no difference was observed in the frequency range of interest ( $0 \leq \omega \leq 10^5$  rad/s), so these systems are chosen to be the reference systems, while the one with the most states (i.e., 62) is chosen to be the exact (correct) system. Closely following the rest of the proposed procedure outlined in Section 4.2.1, the results of the final iteration—i.e., when the feasible solution is obtained and robust performance confirmed—are shown in Table 5.4.

**Table 5.2**

Results for case # 1. Results after the feasible solution in the IQC robustness analysis is found. Different number of beams ( $n_b$ ) are considered. Acronyms are used for the resulting interconnected systems: RO (Reduced Order with initially calculated weights  $W_{j,1}$ ), RW (reduced order with Refined Weights  $W_{j,1e}$ ) and SW (reduced order with Static Weights  $W_{j,1s}$ ). The McMillan degree  $v$  of basis-function needed to achieve solution is also displayed. The numerical simulations were carried out on a Linux 64bit machine with Dual Core Intel i5-3317U 2.6GHz and 8Gb RAM, in Matlab version 9.11 and IQCLab Toolbox V3.0.

<b>Case # 1 <math>n_b = 10</math></b>	
discretization number $n_j$ per beam <sup>2</sup>	[6, 6, 5, 4, 4, 3, 3, 3, 2, 1]
orders of reduced order models $m_j$ per beam	[4, 3, 3, 2, 2, 2, 1, 1, 1, 1]
orders $o_j$ of initial weights $W_{j,1}$ per beam	[3, 3, 2, 2, 2, 2, 1, 1, 1, 1]
orders $o_{j,e}$ of refined weights $W_{j,1e}$ per beam	[3, 3, 2, 2, 2, 2, 2, 2, 2, 1]
number of: inputs $\times$ outputs, states of [reference] and (reduced order) system, and decision variables to the IQC COP <sup>3</sup>	22 $\times$ 22 [433] RO: (92) 4280 <sub><math>v=0</math></sub> RW: (92) 4280 <sub><math>v=0</math></sub> SW: (32) 530 <sub><math>v=0</math></sub>
calculation times of the IQC convex optimization problem LMlab + mincx (YALMIP + MOSEK)	RO: 1337.8s (39.4s) RW: 1199.2s (38.8s) SW: 8.5s (4.6s)
induced $\mathcal{L}_2$ -gains of the [nominal system] <sup>4</sup> , the best achievable $\gamma$ (worst case gain using $\mu$ -tools) for	RO: [0.0643] 0.06462 (0.0657) RW: [0.0643] 0.06593 (0.0657) SW: [0.0643] 0.06594 (0.0657)
obtained [robust stability margins using $\mu$ -tools] <sup>5</sup> and ( $v$ -gaps) <sup>6</sup> for	RO: [5.5192] 0.0118 RW: [992.4105] 0.0118 SW: [992.4107] 0.0118

The case was practically unable to perform due to RAM limitations, because number of decision variables was roughly 35k for the RO model. <sup>2</sup> The discretization orders were chosen manually with the premise that systems further from the performance channel, in general, can be discretized using lower orders. <sup>3</sup> Convex optimization problem (COP). <sup>4</sup> Nominal systems are the corresponding  $\Gamma$ . <sup>5</sup> A robust stability margin greater than 1 means that the system is robustly stable for all values of its modelled uncertainty. <sup>6</sup> Calculated  $v$ -gaps represent a measure of the robust stability for interconnected systems that are to be controlled with a closed loop controller.  $v$ -gap close to zero indicates good robustness.

**Table 5.3**

Results for case # 2. Results after the feasible solution in the IQC robustness analysis is found. Different number of beams ( $n_b$ ) are considered. Acronyms are used for the resulting interconnected systems: RO (Reduced Order with initially calculated weights  $W_{j,1}$ ), RW (reduced order with Refined Weights  $W_{j,1e}$ ) and SW (reduced order with Static Weights  $W_{j,1s}$ ). The McMillan degree  $v$  of basis-function needed to achieve solution is also displayed. The numerical simulations were carried out on a Linux 64bit machine with Dual Core Intel i5-3317U 2.6GHz and 8Gb RAM, in Matlab version 9.11 and IQCLab Toolbox V3.0.

<b>Case # 2 <math>n_b = 7</math></b>	
discretization number $n_j$ per beam <sup>2</sup>	[10, 9, 8, 7, 6, 4, 3]
orders of reduced order models $m_j$ per beam	[6, 3, 3, 2, 2, 2, 2]
orders $o_j$ of initial weights $W_{j,1}$ per beam	[4, 3, 2, 2, 2, 2, 2]
orders $o_{j,e}$ of refined weights $W_{j,1e}$ per beam	[3, 3, 2, 2, 2, 1, 1]
number of: inputs $\times$ outputs, states of [reference] and (reduced order) system, and decision variables to the IQC COP <sup>3</sup>	16 $\times$ 16 [387] RO: (88) 6791 <sub><math>v=1</math></sub> RW: (88) 6791 <sub><math>v=1</math></sub> SW: (36) 668 <sub><math>v=0</math></sub>
calculation times of the IQC convex optimization problem LMlab + mincx (YALMIP + MOSEK)	RO: 55822s (123s) RW: 18824.24s (35.6s) SW: 33.6s (4.4s)
induced $\mathcal{L}_2$ -gains of the [nominal system] <sup>4</sup> , the best achievable $\gamma$ (worst case gain using $\mu$ -tools) for	RO: [0.0670] 0.06969 (0.0671) RW: [0.0670] 0.06695 (0.0670) SW:[0.0670] 0.06696 (0.0670)
obtained [robust stability margins using $\mu$ -tools] <sup>5</sup> and ( $\nu$ -gaps) <sup>6</sup> for	RO: [3.7588] $3.1686 \times 10^{-5}$ RW: [159.7053] $3.1686 \times 10^{-5}$ SW: [91.2962] $3.1690 \times 10^{-5}$

The case was practically unable to perform due to RAM limitations, because number of decision variables was roughly 35k for the RO model. <sup>2</sup> The discretization orders were chosen manually with the premise that systems further from the performance channel, in general, can be discretized using lower orders. <sup>3</sup> Convex optimization problem (COP). <sup>4</sup> Nominal systems are the corresponding  $\Gamma$ . <sup>5</sup> A robust stability margin greater than 1 means that the system is robustly stable for all values of its modelled uncertainty. <sup>6</sup> Calculated  $\nu$ -gaps represent a measure of the robust stability for interconnected systems that are to be controlled with a closed loop controller.  $\nu$ -gap close to zero indicates good robustness.

**Table 5.4**

Results for case # 3. Results after the feasible solution in the IQC robustness analysis is found. Different number of beams ( $n_b$ ) are considered. Acronyms are used for the resulting interconnected systems: RO (Reduced Order with initially calculated weights  $W_{j,1}$ ), RW (reduced order with Refined Weights  $W_{j,1e}$ ) and SW (reduced order with Static Weights  $W_{j,1s}$ ). The McMillan degree  $v$  of basis-function needed to achieve solution is also displayed. The numerical simulations were carried out on a Linux 64bit machine with Dual Core Intel i5-3317U 2.6GHz and 8Gb RAM, in Matlab version 9.11 and IQCLab Toolbox V3.0.

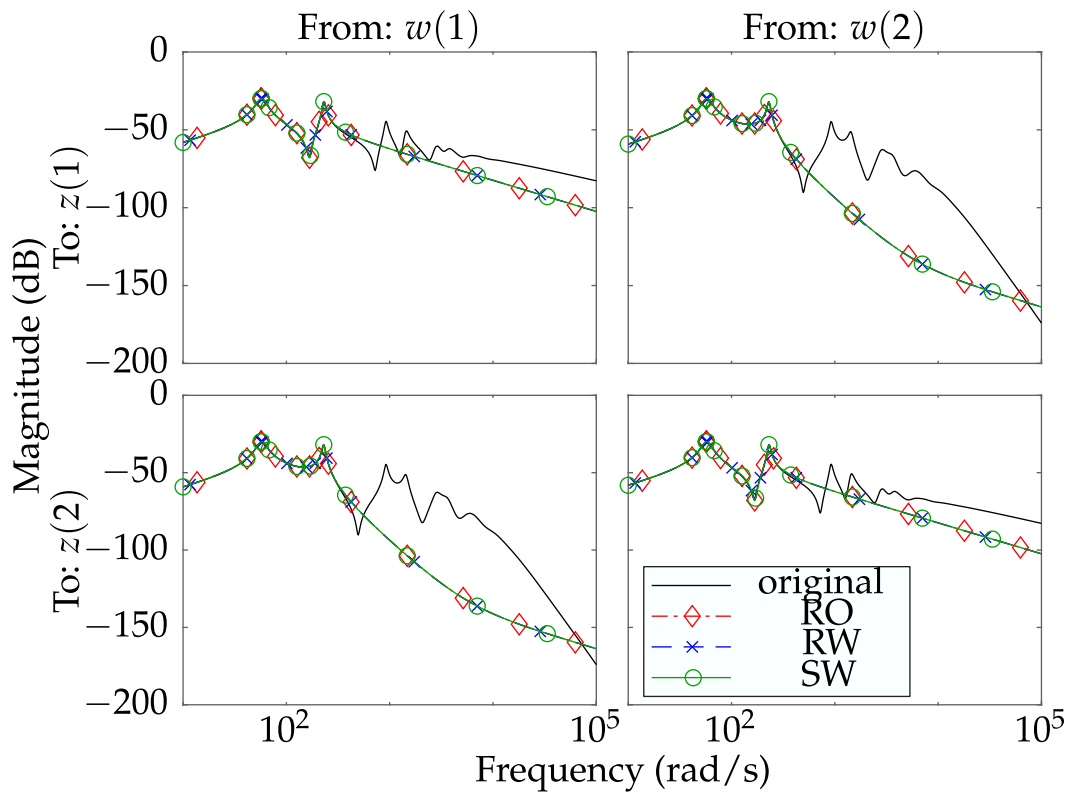
<b>Case # 3 <math>n_b = 5</math></b>	
discretization number $n_j$ per beam <sup>2</sup>	[10, 10, 10, 10, 10] <sup>1</sup>
orders of reduced order models $m_j$ per beam	[6, 3, 3, 2, 2]
orders $o_j$ of initial weights $W_{j,1}$ per beam	[4, 3, 2, 2, 2]
orders $o_{j,e}$ of refined weights $W_{j,1e}$ per beam	[4, 3, 2, 2, 2]
number of: inputs $\times$ outputs, states of [reference] and (reduced order) system, and decision variables to the IQC COP <sup>3</sup>	12 $\times$ 24 [310] RO: (120), n/a RW: (120), n/a SW: (48) 5896 <sub><math>v=2</math></sub>
calculation times of the IQC convex optimization problem LMilab + mincx (YALMIP + MOSEK)	RO: n/a RW: n/a SW: 10600.6 (113.2s)
induced $\mathcal{L}_2$ -gains of the [nominal system] <sup>4</sup> , the best achievable $\gamma$ (worst case gain using $\mu$ -tools) for	RO: [4.6115] n/a (4.6324) RW: [4.6115] n/a (4.6124) SW: [4.6115] 4.622 (4.6138)
obtained [robust stability margins using $\mu$ -tools] <sup>5</sup> and ( $\nu$ -gaps) <sup>6</sup> for	RO: [2.7540] 0.0263 RW: [348.1343] 0.0263 SW: [80.8150] 0.0265

<sup>1</sup> The case was practically unable to perform due to RAM limitations, because number of decision variables was roughly 35k for the RO model. <sup>2</sup> The discretization orders were chosen manually with the premise that systems further from the performance channel, in general, can be discretized using lower orders. <sup>3</sup> Convex optimization problem (COP). <sup>4</sup> Nominal systems are the corresponding  $\Gamma$ . <sup>5</sup> A robust stability margin greater than 1 means that the system is robustly stable for all values of its modelled uncertainty. <sup>6</sup> Calculated  $\nu$ -gaps represent a measure of the robust stability for interconnected systems that are to be controlled with a closed loop controller.  $\nu$ -gap close to zero indicates good robustness.

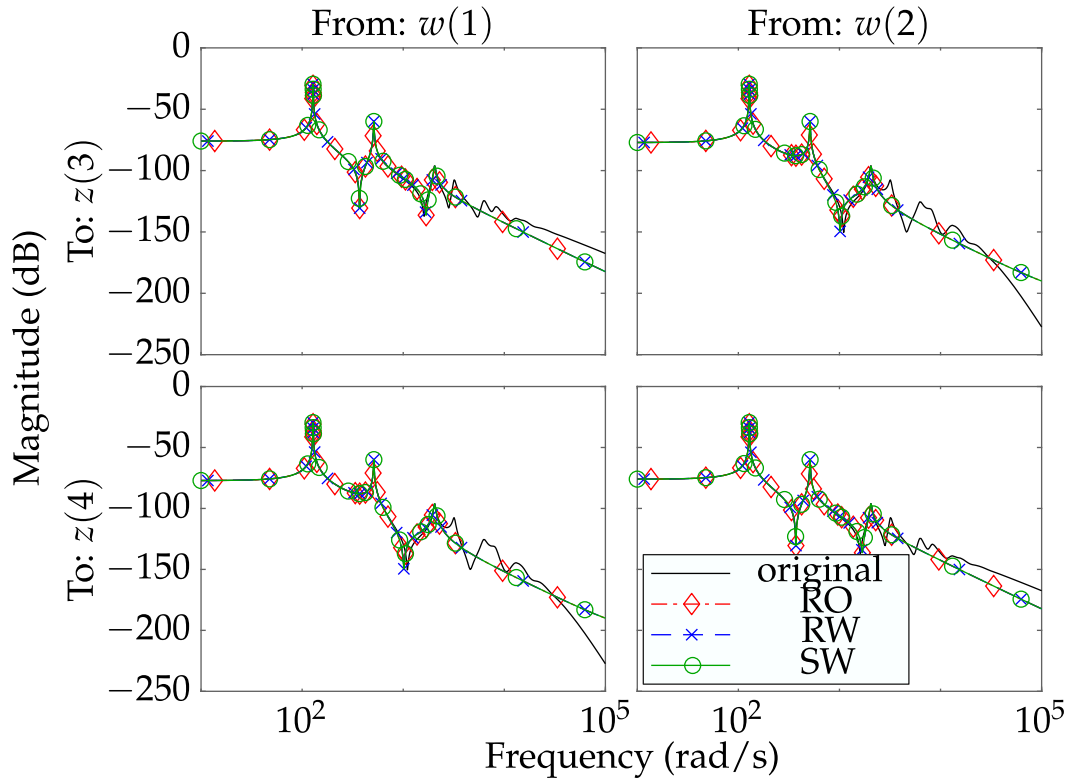
An interesting observation can be made—for the systems near the external inputs and outputs discretization level, as well as the order of the reduced order model, needs to be higher when compared to the systems that are further away from the external inputs and outputs. A physical explanation for this phenomenon is that most of the dynamics for those systems (i.e., further away) is dissipated thorough the system and as such these systems have small influence on the overall dynamic response on the

channel  $w \rightarrow z$ —thus their dynamics can be chosen to be of lower order. This is especially true for higher order dynamics that naturally get damped relatively fast.

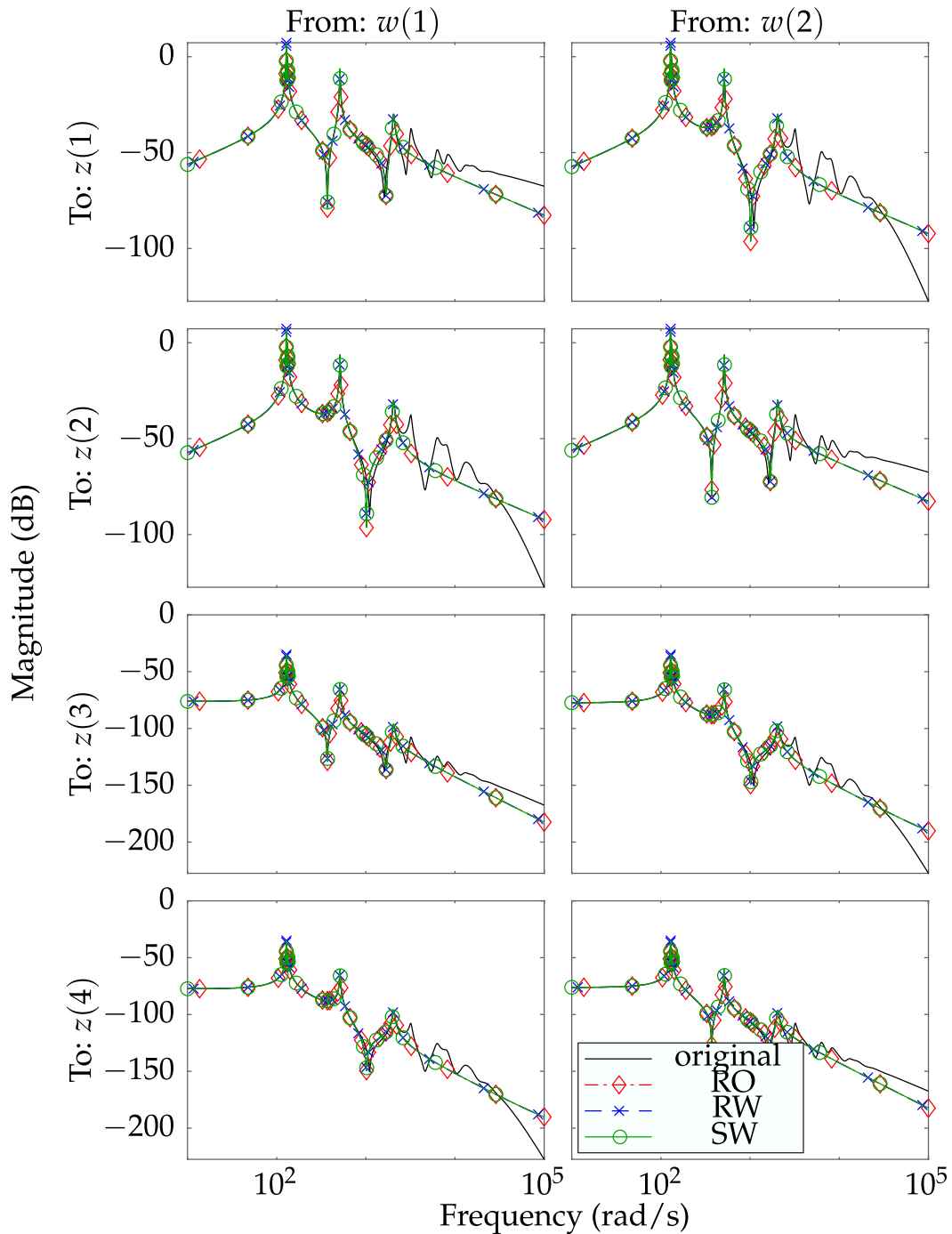
In Figures 5.2, 5.3 and 5.4, it can be seen that the frequency response is highly correlated in the lower frequency range with acceptable discrepancy in higher frequency range. Despite noticeably lower order for the interconnected system that uses the static weights  $W_{j,1s}$  (SW), the results are basically the same as the other two reduced order systems (i.e., RO and RW). From Table 5.4, the best achievable induced  $\mathcal{L}_2$ -gains are very well correlated with the unreduced interconnected system. The same can be observed for the results obtained using  $\mu$ -tools and for the  $\nu$ -gaps. These results confirm the achieved robustness using IQC analysis and IQCLab toolbox.



**Figure 5.2:** Case # 1 — Frequency response of the original (unreduced) interconnected system compared to RO (Reduced Order with initially calculated weights  $W_{j,1}$ ), RW (reduced order with Refined Weights  $W_{j,1e}$ ) and SW (reduced order with Static Weights  $W_{j,1s}$ ). Case # 1, beams mutually interconnected only by dampers, representing a highly dissipative case.



**Figure 5.3:** Case # 2 — Frequency response of the original (unreduced) interconnected system compared to RO (Reduced Order with initially calculated weights  $W_{j,1}$ ), RW (reduced order with Refined Weights  $W_{j,1e}$ ) and SW (reduced order with Static Weights  $W_{j,1s}$ ). Case # 2, beams mutually interconnected only by springs, representing the least dissipative (stiffest) case.



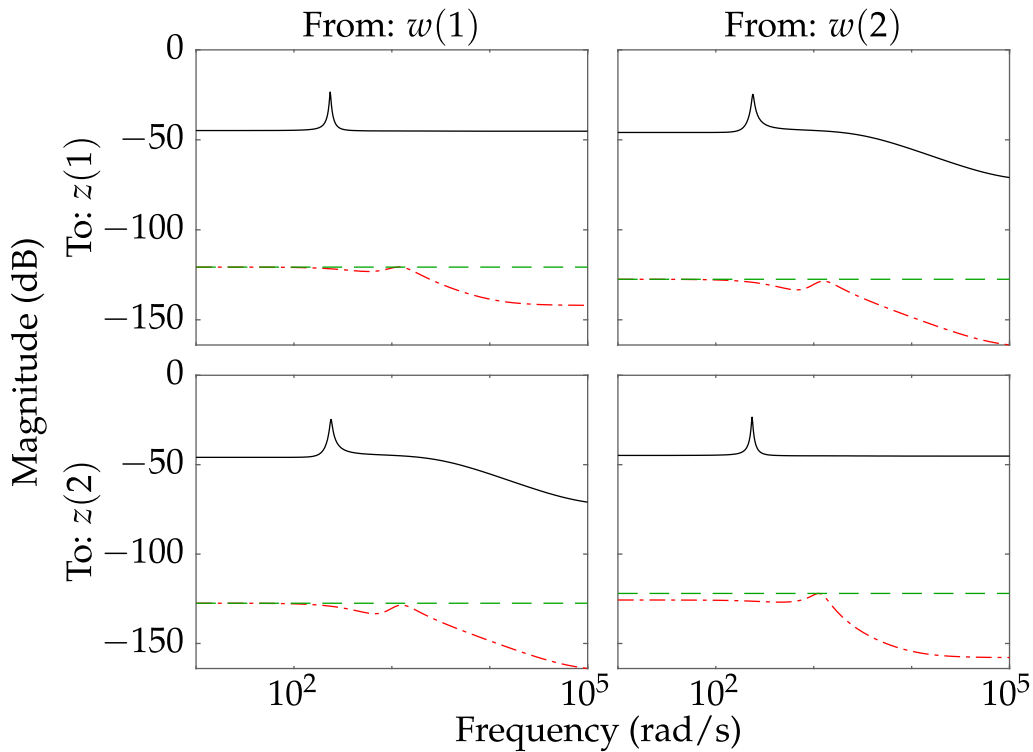
**Figure 5.4:** Case # 3 — Frequency response of the original (unreduced) interconnected system compared to RO (Reduced Order with initially calculated weights  $W_{j,1}$ ), RW (reduced order with Refined Weights  $W_{j,1e}$ ) and SW (reduced order with Static Weights  $W_{j,1s}$ ). Case # 3, beams mutually interconnected by both dampers and springs, representing the most complex interconnections case (with most uncertainty channels).

When the obtained weights are compared, as seen in Figures 5.5 to 5.13, it can be seen in the first row of results (for case # 1), the refined uncertainty can easily be chosen as static and perhaps even discarded altogether as it has low gain. Case # 1 represents

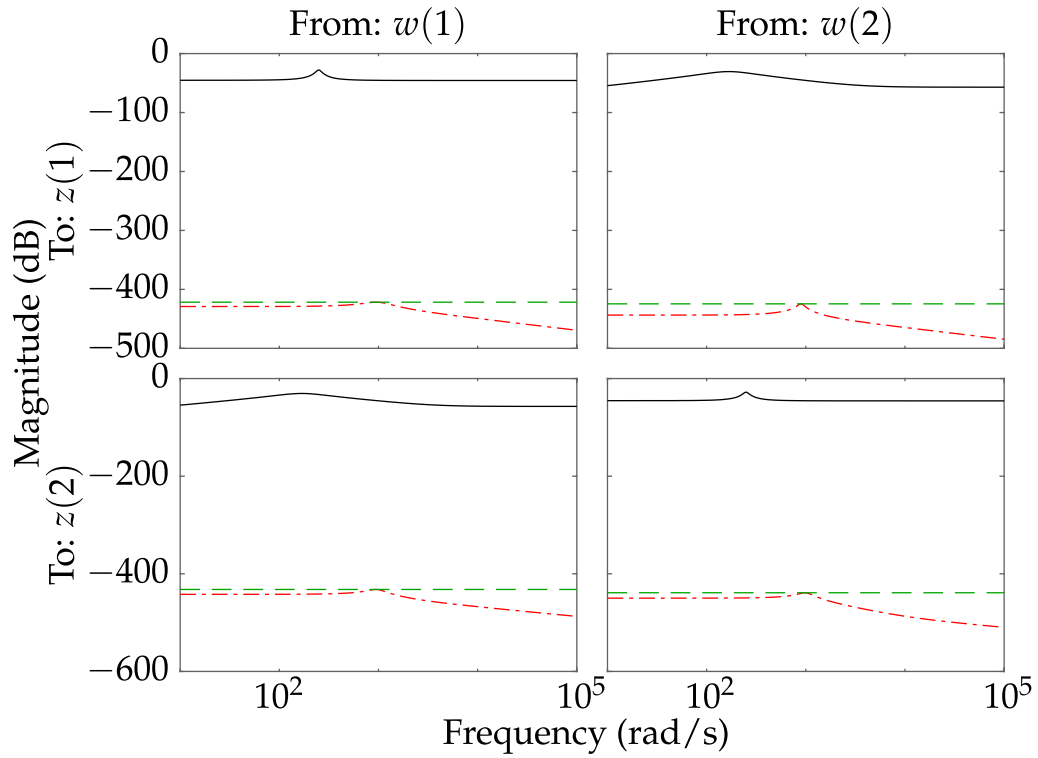


a passive system and this property might be additionally exploited for even better results [27]. The greatest challenge was to obtain suitable weights for the system that has two inputs and four outputs per subsystem, i.e., case # 3. All the IQC robustness analyses shown that it is possible to use static filters (model SW) for guaranteed robust stability and robust performance.

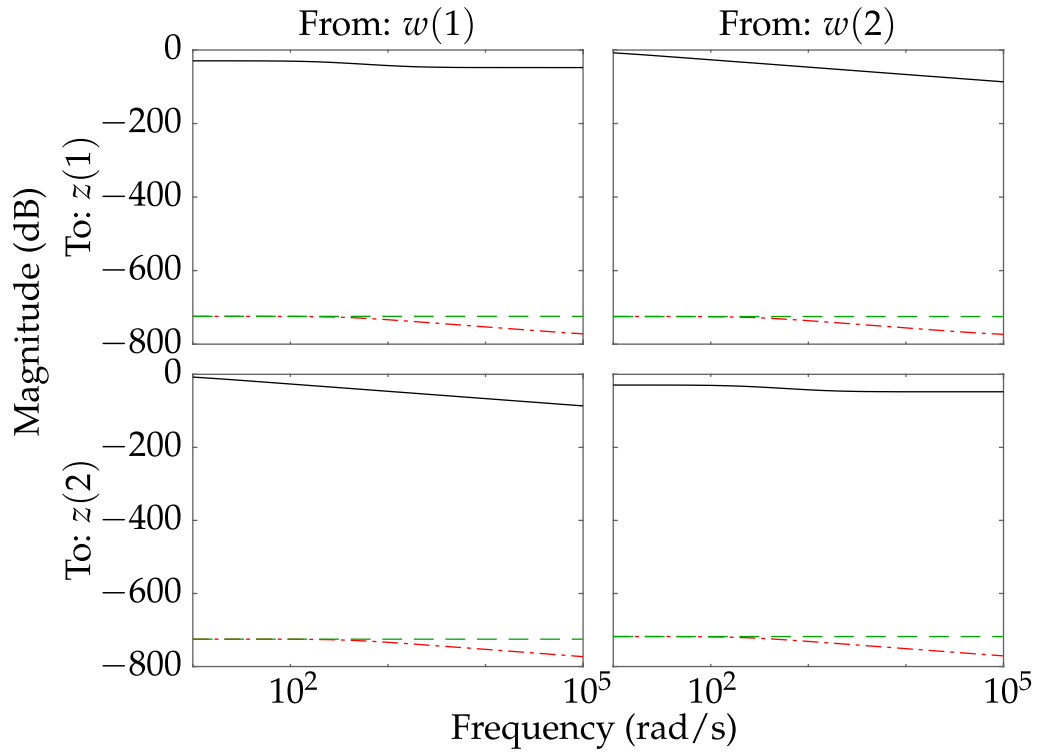
Results obtained on the previously compared cases should be considered only representative. It is important to point out that this is a modelling technique. As such, it is dependent on the peculiarities of the problem at hand. The choice for discretization level  $n_j$ , reduced model order  $m_j$ , as well as the order for the obtained weights  $o_{j,e}$  (that define a static weight  $W_{j,1s}$ ), per subsystem, at first might seem completely heuristic. However, author stresses that these choices can be chosen intuitively and even leveraged as an advantage, if more precise modelling criteria are given. To illustrate what the author means by this, let us reconsider Case #1.



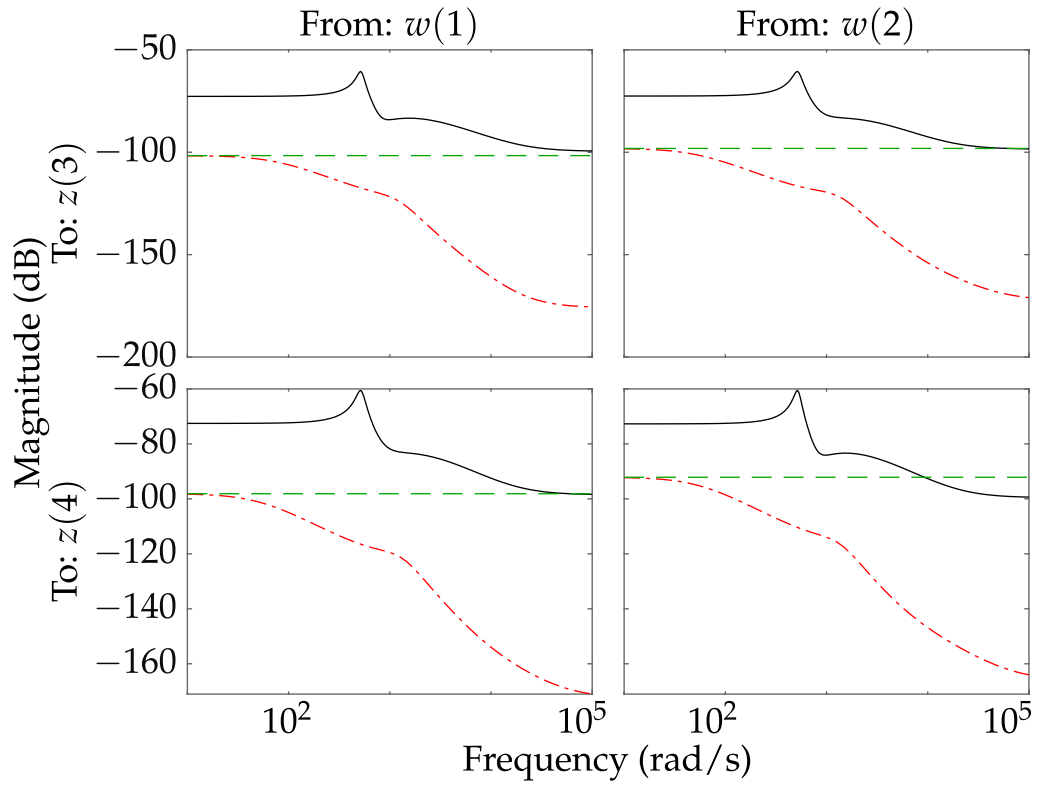
**Figure 5.5:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 1, system 3.



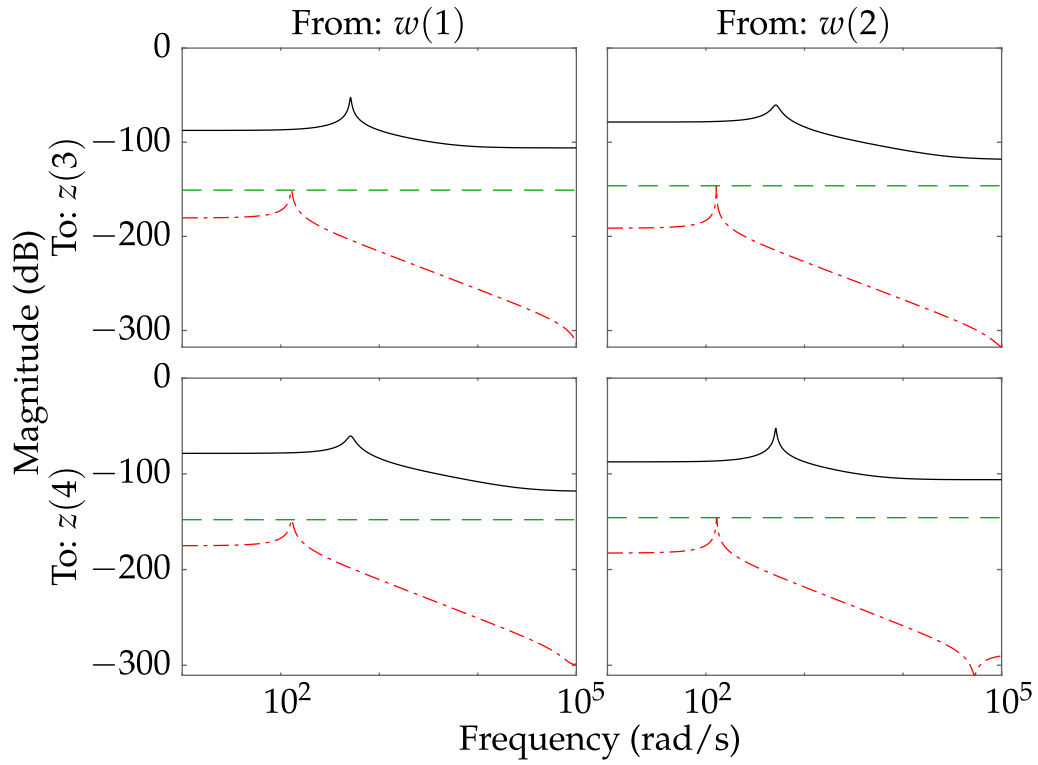
**Figure 5.6:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 1, system 11.



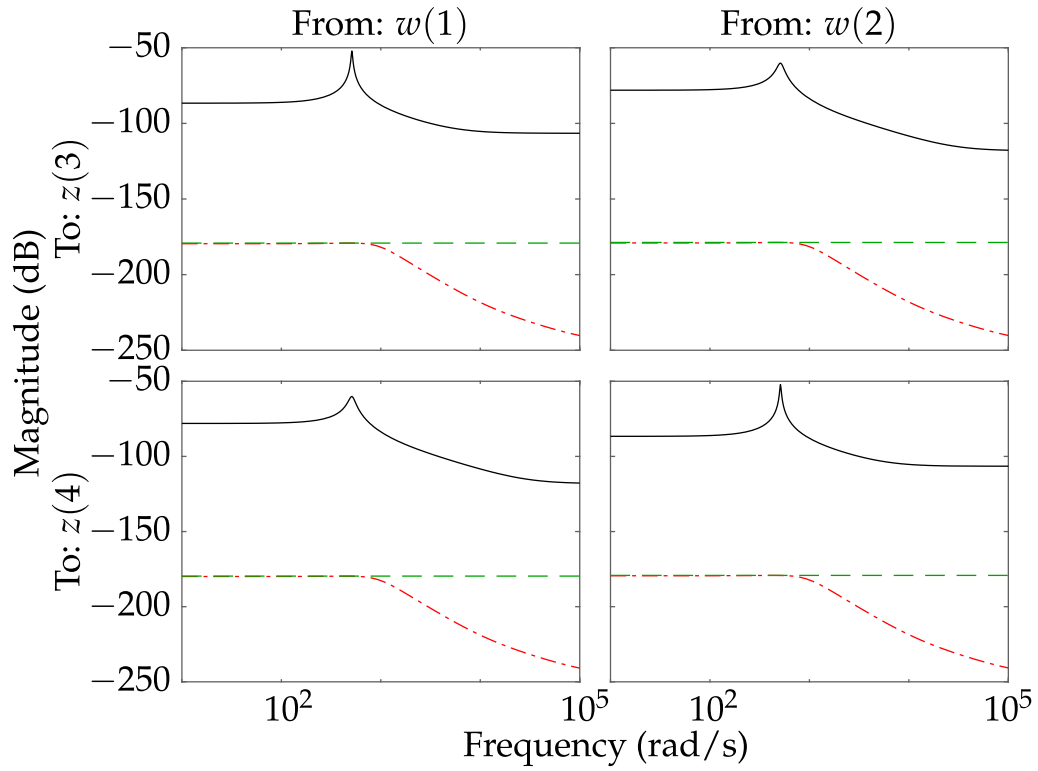
**Figure 5.7:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 1, system 19.



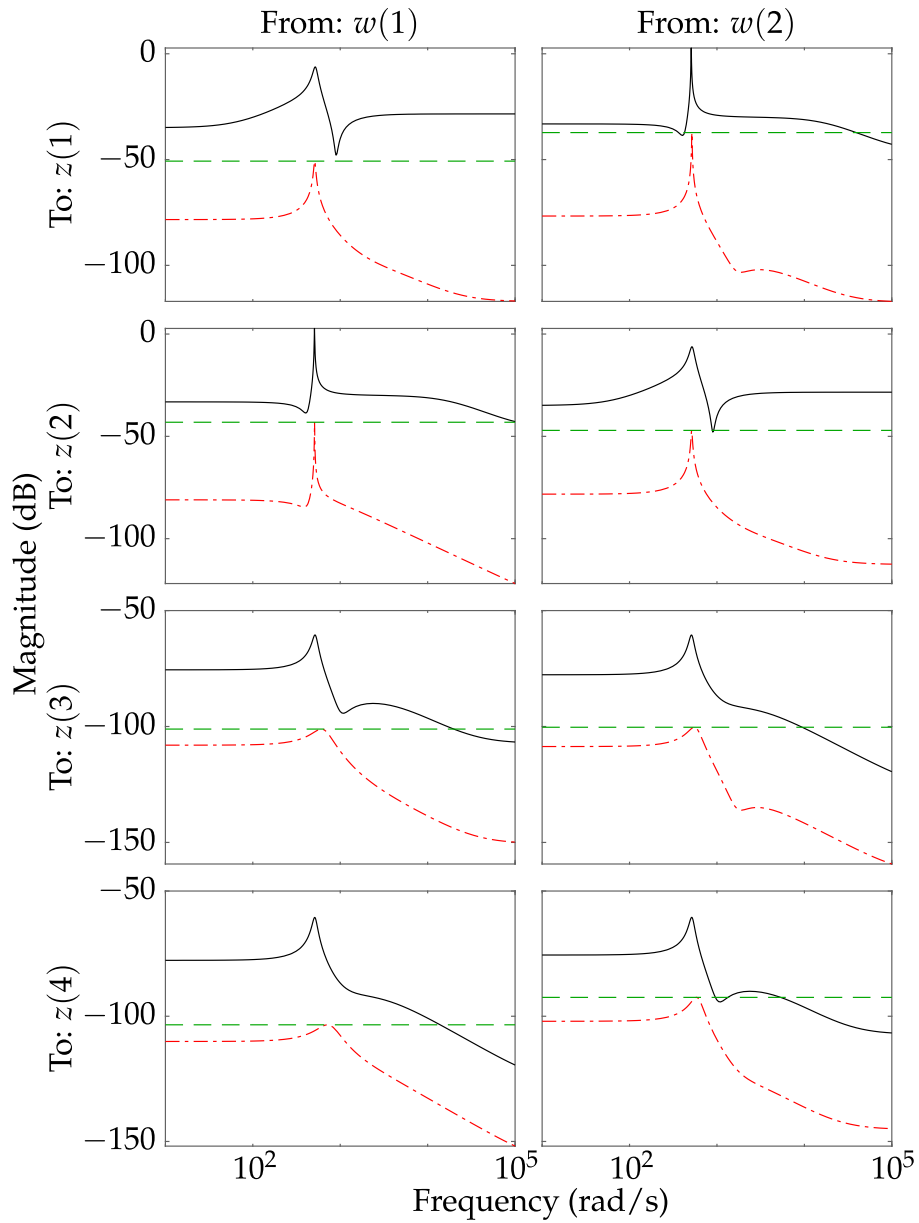
**Figure 5.8:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 2, system 3.



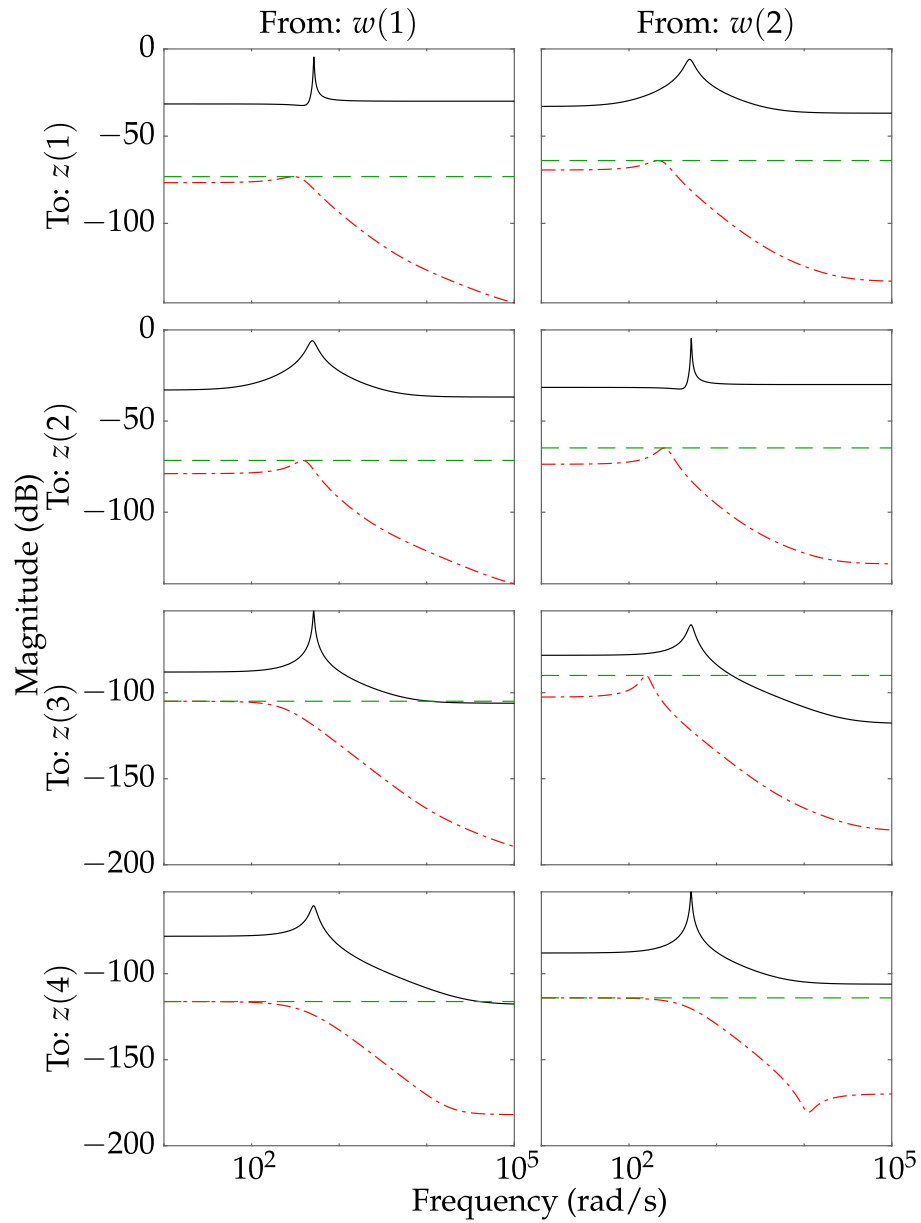
**Figure 5.9:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 2, system 7..



**Figure 5.10:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 2, system 13.

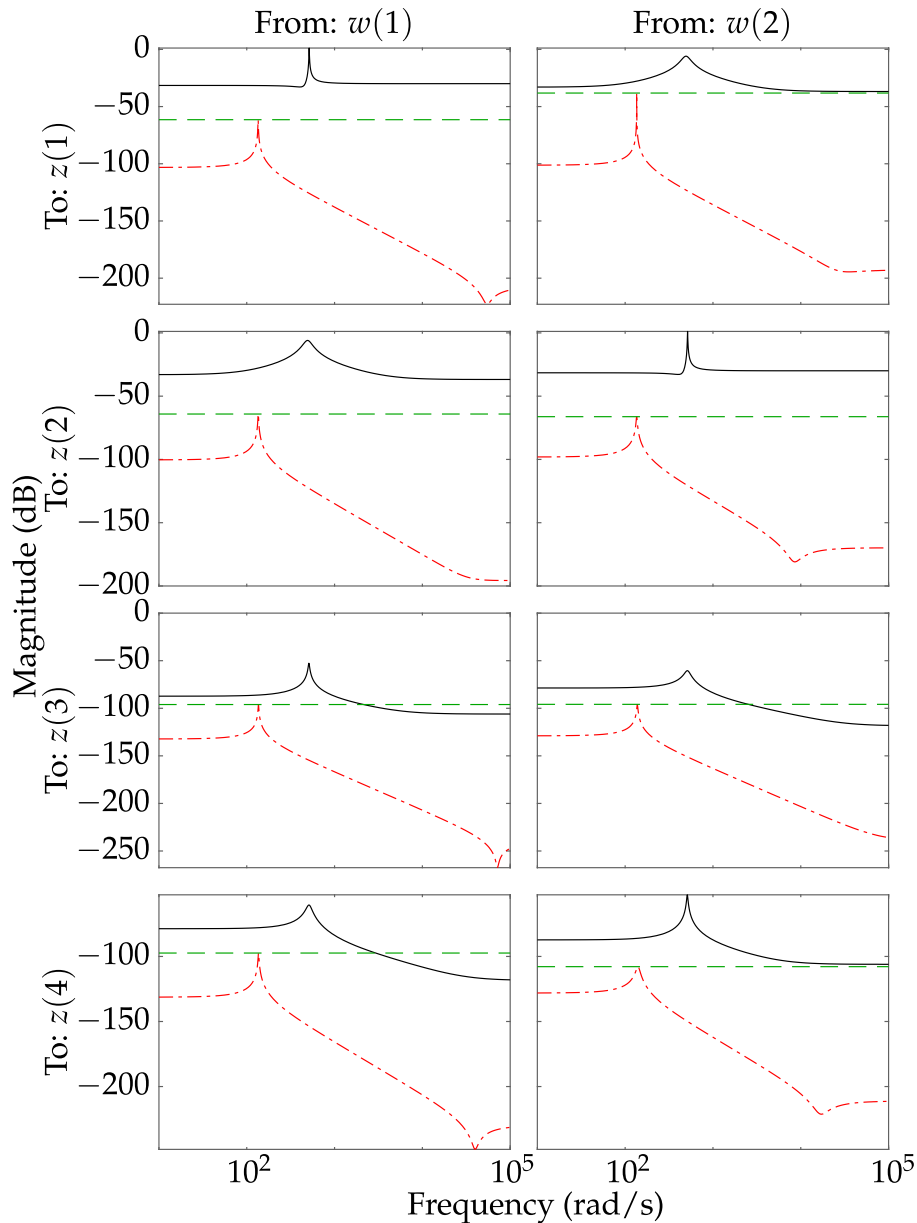


**Figure 5.11:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 3, system 3.



**Figure 5.12:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 3, system 5.

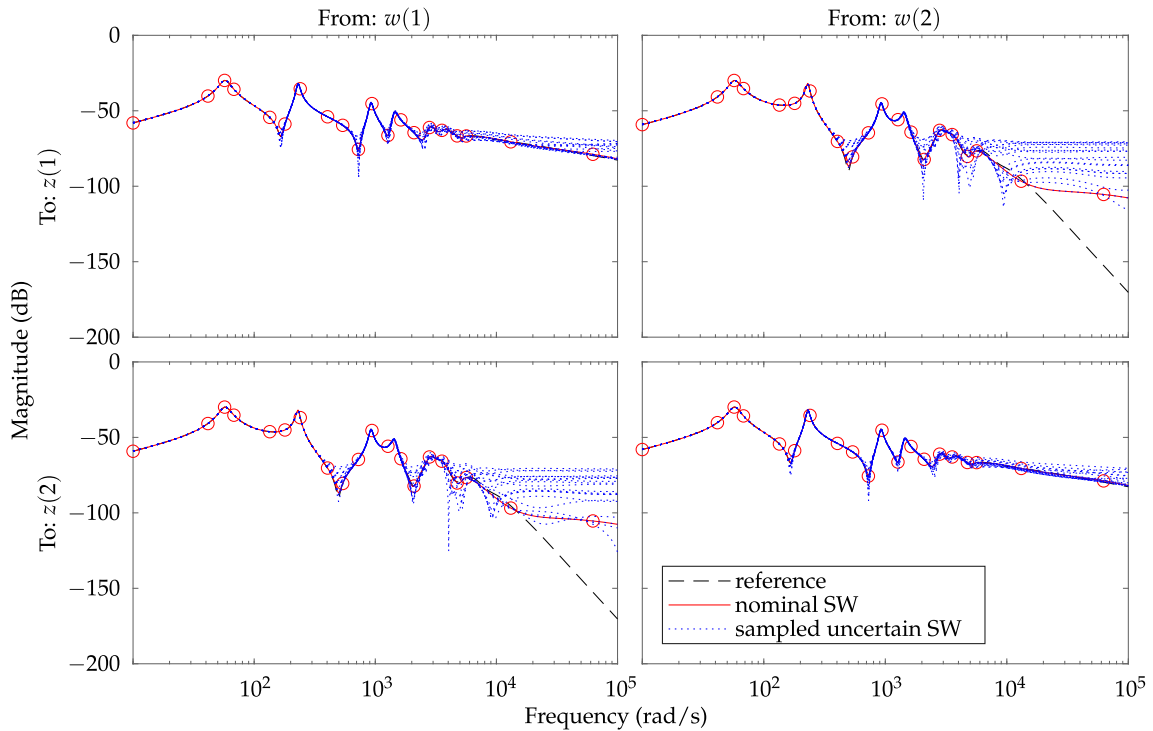




**Figure 5.13:** Frequency response of the with initially calculated weights  $W_{j,1}$  (solid black line), refined weights  $W_{j,1e}$  (dash-dot red line) and static weights  $W_{j,1s}$  (dashed green line). Case # 3, system 9.

Let us assume that the divergence in the higher frequencies is not meeting the desired criteria, despite the obtained system met the robustness criteria. Let us further assume that the order of the resulting reduced order (uncertain) interconnected system is satisfactory and that there is room for a slight increase in the overall order of the interconnected system. Thus, a better correlation in the broader frequency range (for this specific example), can be achieved with an increase in reduced model order for the first subsystem. One could also argue that most of the uncertainty, for this specific case, can be captured by only modelling uncertainty for the few systems near the external inputs and outputs. By observing Figures 5.5 to 5.13 it can be concluded that

the amount of refined uncertainty for the beams number 2 to 10 is negligible (gain of around  $-150$  dB or lower). As such, one can completely discard the uncertainty for these systems and only keep static weight for the first system. As a results, one can try to carry out a robustness analysis for a reduced order (uncertain) interconnected system that has  $n_j = [10, 6, 5, 2, 2, 2, 2, 1, 1, 1]_{n_b=10}$ ,  $m_j = [18, 3, 2, 1, 1, 1, 1, 1, 1, 1]_{n_b=10}$  and  $o_{j,e} = [3, -, -, -, -, -, -, -, -, -]$  (dashes  $(-)$  indicate no uncertainty is modeled for a particular subsystem). Frequency response of the newly obtained interconnected system is shown in Figure 5.14.



**Figure 5.14:** Case #1 with different  $n_j$  and  $m_j$ . Using higher discretization and higher order for the first subsystem, results in a better correlation in the frequency response when compared to the original system. Few random uncertainty samples are shown to give an idea of the frequency response of the uncertain interconnected system.

Indeed, one does obtain a feasible solution to the IQC analysis (and a robust stability margin greater than one, having value of  $6.7219 \times 10^4$ ) confirming the system is robustly stable. One can also achieve  $\gamma = 0.064511$ , and as such being close to a nominal value of  $0.0643$  (and also confirmed using  $\mu$ -tools by calculating worst case gain that has a value of  $0.0645$ ). The  $\nu$ -gap (when compared to the reference model) also drastically decreased to a value of  $2.9522 \times 10^{-4}$ , further demonstrating that a newly obtained uncertain dynamical system is close to a reference system—if the systems are to be controlled by a same controller—indicating good robustness in a closed loop scenario.

A better correlation when compared to the reference (original) system is now evident

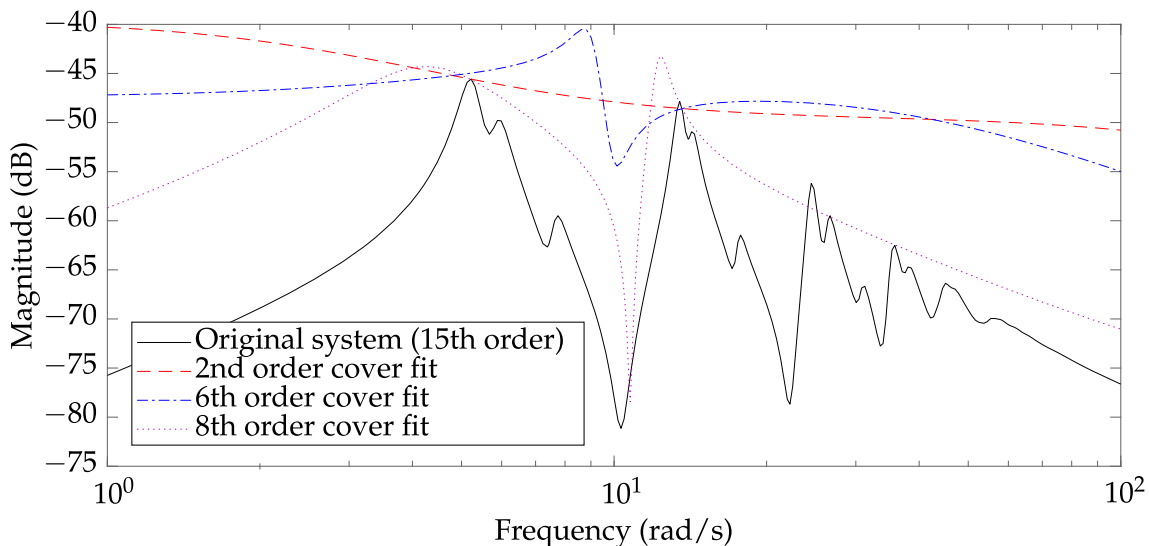
up to roughly  $10^4$  rad/s can be seen in Figure 5.14 (when compared to case #1 in Figures 5.5 to 5.7, there was good correlation only up to roughly  $10^2$  rad/s). To get a sense of the overall uncertain behaviour of the system, few random samples of the uncertain model are plotted as well—showing an (expected) increase in the uncertainty in the higher frequencies due to discretization and reduced order subsystems. It should be noted that the obtained results are still sub-optimal, but suitable for the specified criteria. Moreover, this case of weakly coupled systems (i.e., beams interconnected with only viscous dampers) might seem extreme, but to a point serve to demonstrate how modelling of a very large scale system (i.e., consisting of thousands of dissipative systems) might be approached. Amount of the required uncertainty, for dynamical systems that are distant from the external inputs and outputs, can often be drastically reduced or in some cases even completely discarded as shown above—this, in turn, results in reducing the conservatism of the uncertain interconnected system.

## 5.2 Discussion

### 5.2.1 On the choice of weight design

Most often, the discrepancy between the nominal and the reduced order discretized model is evident in the high frequency range. One simple way to scale the uncertainty with the frequency weights  $W_{j,1}$  and  $W_{j,2}$ , is to model them as high-pass filters. Perhaps a bit more sophisticated and robust design is to use logarithmic-Chebyshev magnitude design, as outlined in [136, 67], that guarantees stability of the weight  $W_{j,1}$ , while also being minimum phase-shaping and can be chosen such that it is low-order and *covers* the modelled absolute error from above in the important frequency range. This gives the advantage of not missing the high frequency dynamics at the expense of modest to none increase in conservatism. It should be noted that there is no single recipe for designing any weight for that matter, and the choice of a weight is often a result of design experience, experimentation or perhaps even obtained via trial and error. Let us examine an example of a high order system that has a frequency response as shown in Figure 5.15. As it can be observed, independent of the order of the filter, its response always lies above the original system, thus making this type of *cover* filter useful for unstructured uncertainty modelling [128, 60]. From Figure 5.15, it can also be seen that the magnitude of the filter that has order two (2nd order cover fit) is larger in wide frequency range than that of filter that has order six (6th order cover fit) and order eight (8th order cover fit). One could imagine that the given original system of order fifteen (15th order) was an absolute error that one wants to model using (unstructured) additive uncertainty and that lower order cover filters (2nd, 6th and 8th order cover fit) are weights  $W_{j,1}$ . For this particular example, a general conclusion

would be that the conservatism of the 8th order filter would thus be lower than the conservatism of 6th order cover filter and especially lower than that of 2nd order cover filter. The explanation for this lies in the fact that the 2nd order filter (when multiplied by a dynamic system of norm less than 1—i.e., unstructured uncertainty) includes more dynamics than the other two that are more closely covering the original dynamical system. In Figure 5.15, it can also be seen that the 8th order cover filter still has significantly larger gain in lower frequencies—thus, the uncertainty in lower frequencies, for this example, would still be conservative. One way to tackle this problem would be to use one more frequency weight (filter)—i.e.,  $W_{j,2}$  multiplying uncertainty from the left—and model it in such a way to further scale the uncertainty in the lower frequencies. In the case of discretized and reduced order models, lower order dynamics is usually correlated well when compared to the higher order models, so in this case, a possible benefit of  $W_{j,2}$  might be to reduce conservatism at higher frequencies.

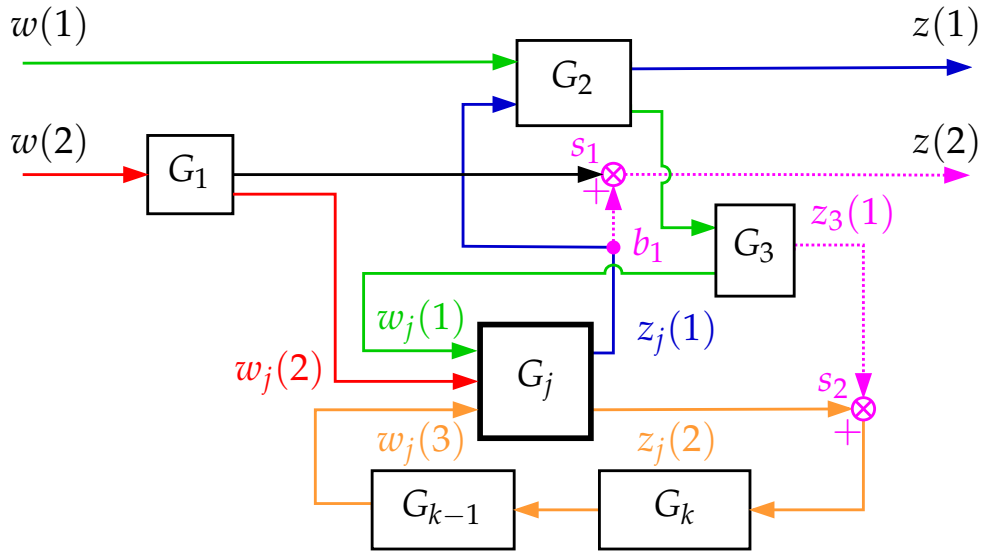


**Figure 5.15:** Examples of the different low-order weights (filters)—modelled using logarithmic-Chebyshev filter design—that *cover* the frequency response of a high order dynamical system in the desired frequency range. All lower order filters are stable and modelled in such a way to have gain larger than the original high order dynamical system. This makes such a filter useful when modelling unstructured uncertainties (e.g., additive uncertainty).

## 5.2.2 Defining the unique paths of energy transfer throughout the system

Let us now consider an example of an interconnected dynamical system, that is assumed to be dissipative, as represented with Figure 5.16. Dissipativity of the interconnected system manifests itself with the dissipativity on the subsystem level and/or through dissipative interconnections between them. For the example at hand, let us

assume that there are two performance (external) inputs  $w = (w(1), w(2))^T$  and two performance (external) outputs  $z = (z(1), z(2))^T$ . Similar vector expansion of internal inputs  $w_j \in \mathcal{L}_2^{n_{w_j}}$  and internal outputs  $z_j \in \mathcal{L}_2^{n_{z_j}}$  is assumed, where  $n_{w_j}$  and  $n_{z_j}$  represent the number of inputs and the number of outputs of the  $G_j$ -th subsystem, respectively. The main idea is to define all the unique paths through which the energy can be transferred. To do so, one can examine the inputs and outputs of each subsystem.



**Figure 5.16:** Example of the interconnected dissipative dynamical system. Defining the paths of energy transfer to and from the subsystem  $G_j$ .

Let us for that purpose focus on the  $j$ -th subsystem  $G_j$  that has three inputs  $w_j = (w_j(1), w_j(2), w_j(3))^T$  and two outputs  $z_j = (z_j(1), z_j(2))^T$ . Let us can start by examining the energy transfer from external inputs  $w$  to the internal inputs  $w_j$ , of the subsystem  $G_j$ , and from internal outputs  $z_j$  to the external outputs  $z$ . In order for the energy to transfer from the external input  $w(1)$  to the first input of the subsystem  $G_j$  it needs to take the path through the subsystems  $G_2$  and  $G_3$ —this is the *green coloured* energy transfer path in Figure 5.16. Following the same procedure for the external input  $w(2)$ , the energy passes through the system  $G_1$ —before landing on the second input of the subsystem  $G_j$ —following the *red coloured* energy transfer path shown in Figure 5.16. On the other hand, if the energy path is followed from the first output of the subsystem  $G_j$ , it can be observed that it needs to pass through the subsystem  $G_2$ , before reaching the external output  $z(1)$ —following the *blue coloured* energy transfer path, as shown in Figure 5.16. It can also be observed that part of the energy is also transferred from the second internal output  $z_j(2)$  of the subsystem  $G_j$  to the the third internal input  $w_r(3)$  of the subsystem  $G_j$ , and doing so passes through the subsystems  $G_k$  and  $G_{k-1}$ —following the *orange coloured* path of energy transfer. It is also possible

to take into account the branching points and summation points together with the accompanying energy transfer paths following the *dashed pink coloured* in Figure 5.16. By doing so, the following can be observed. Some of the energy is directly transferred from subsystem output  $z_j(1)$  through the branching point  $b_1$  and the summation point  $s_1$  to the external output  $z(2)$ . Note, in this case, although the inputs to the summation point  $s_1$  are  $z_1(2)$  and  $z_j(1)$ , one is only interested in the energy transferred from the system at hand to the external output  $z(2)$ . Then again, some of the energy coming from the external input  $w(1)$  passes through the subsystems  $G_2$  and  $G_3$  before arriving at the summation point  $s_2$  and transferring energy through the subsystems  $G_k$  and  $G_{k-1}$  towards the internal input  $w_j(3)$  of the subsystem  $G_j$ . In this case, since some of the energy from the external input  $w(1)$  is being dissipated, one does include this channel at the summation point  $s_2$ .

### 5.2.3 Replacing the surroundings of a subsystem with input-output transfer functions

After all the distinct paths of energy transfer for the system at hand (i.e.,  $G_j$  in this case) are obtained, one can calculate the input-output transfer functions (IOTF) for each path of energy transfer. By doing so, it is possible to find the minimum number of systems that will be connected directly to the internal inputs and outputs of the subsystem at hand, replacing the complex structure of the overall environment of one particular subsystem at hand. To simplify the explanation and calculation for the example shown in Figure 5.16, let us examine the calculation in the complex frequency domain ( $s$ -domain) with signals  $\hat{w}$ ,  $\hat{w}_j$ ,  $\hat{z}$  and  $\hat{z}_j$  by taking the Laplace transform of external inputs signals  $w$ , internal inputs signals  $w_j$ , external outputs signals  $z$  and internal outputs signal  $z_j$ , respectively. For the case of LTI systems, calculation for the IOTF of the subsystem  $G_j$  can be carried out as

$$G_{w(1) \rightarrow w_j(1)} = \frac{\hat{w}_j(1)}{\hat{w}(1)} = \hat{G}_3 \hat{G}_2, \quad (5.1)$$

$$G_{w(2) \rightarrow w_j(2)} = \frac{\hat{w}_j(2)}{\hat{w}(2)} = \hat{G}_1, \quad (5.2)$$

$$G_{z_j(1) \rightarrow z(1)} = \frac{\hat{z}(1)}{\hat{z}_j(1)} = \hat{G}_2, \quad (5.3)$$

$$G_{z_j(2) \rightarrow w_j(3)} = \frac{\hat{w}_j(3)}{\hat{z}_j(2)} = \hat{G}_{k-1} \hat{G}_k, \quad (5.4)$$

$$G_{z_j(1) \rightarrow z(2)} = \frac{\hat{z}(1)}{\hat{z}_j(1)} = I, \quad (5.5)$$

$$G_{w(1) \rightarrow z_3(1)} = \frac{\hat{z}_3(1)}{\hat{w}(1)} = \hat{G}_3 \hat{G}_2, \quad (5.6)$$

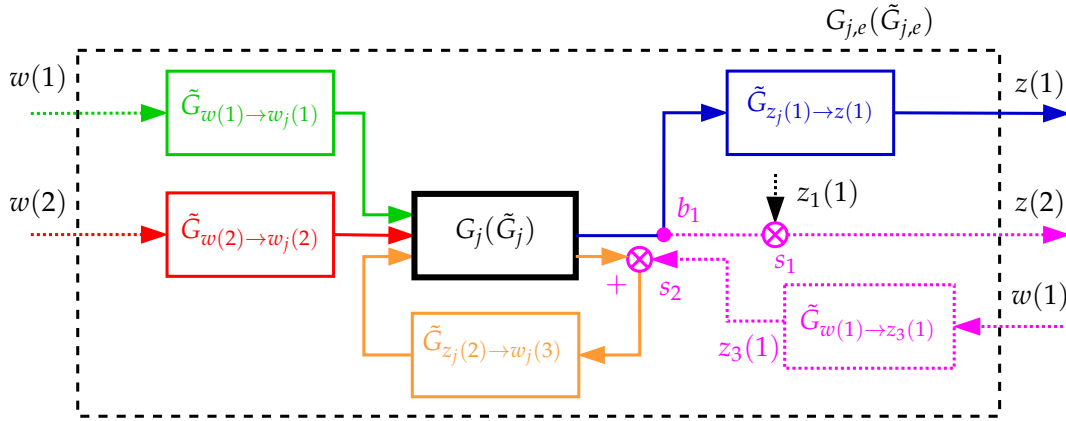
where  $\hat{G}_1, \hat{G}_2, \hat{G}_3, \hat{G}_{k-1}$  and  $\hat{G}_k$  are transfer functions of  $G_1, G_2, G_3, G_{k-1}$  and  $G_k$ , respectively.

The summation points  $s_1$  and  $s_2$  are defined as algebraic relations

$$z(2) = z_j(1) + z_1(1) \tag{5.7}$$

$$w_k(1) = z_j(2) + z_3(1). \tag{5.8}$$

The interconnected system defined by Equations (5.1)–(5.8) can be represented as shown in Figure 5.17. For such an interconnected system, interconnection matrices can be made using the same procedure as outlined in Section 2.3. To make a distinction between the interconnection matrices for interconnecting the whole system, which from now on will be called global interconnection matrices, these will simply be called local IOTF connection matrices.



**Figure 5.17:** Obtaining the composite subsystems  $G_{j,e}$  and  $\tilde{G}_{j,e}$ , scaled by its environment, by connecting the reduced order input-output transfer functions (IOTF) to the inputs and outputs of the examined subsystem  $G_j$  or reduced order subsystem  $\tilde{G}_j$ , respectively.

### On the calculation of the IOTFs and its practical applicability

Although the example shown in Figure 5.16 had taken into consideration many possible scenarios of interconnections, this part remains to be explored for other combinations of interconnection. For this particular example, all the external inputs and outputs are highly clustered together (i.e., in the top part of the interconnected system) in a sense that some external signal is injected close to where it is measured—the example of this might be active car suspension, i.e., measurements include the tyre kinematics while forces are acting on the active suspension, and car body representing the overall distant part of the system. Then there might be systems that have external inputs on one side and at a distant side the outputs are measured—for these kinds

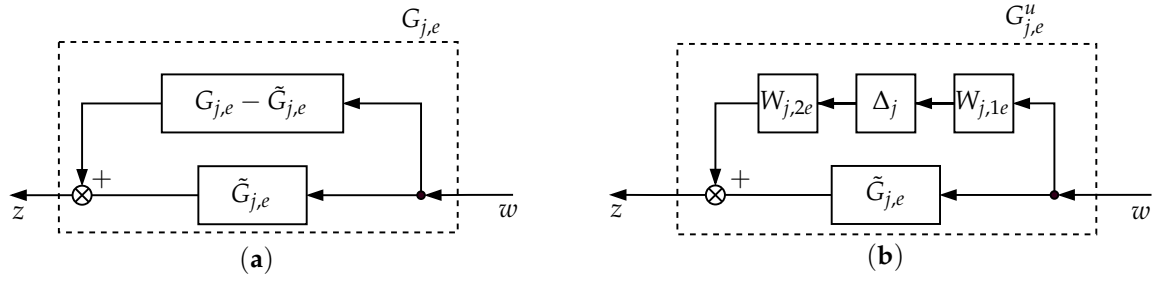
of systems the procedure is practically the same, and will, arguably, also result in relatively small number of IOTF systems. A practical example of such a system might include active cruise control, i.e., measurements are taken at the tyre, some energy is carried throughout the power-train system and some action is taken on the engine side. However, if the external inputs and outputs are highly dispersed—i.e., everywhere in the system there are external inputs and everywhere in the system measurements are taken—this procedure might not yield the expected results and might not give major practical advantage over regular non-preserving methods. It should be noted that this type of input-output arrangement is not a typical case in practical application of the control. Moreover, finding all the energy transfer paths for all the subsystems might be cumbersome. Although there might not be fully automatic way to create IOTF, there are ready made solutions that greatly improve the applicability as it will be mentioned in the numerical example. It should also be noted that, if some of the energy paths are omitted, the only consequence will be a (slightly) more conservative uncertainty model [60, 67, 128].

#### 5.2.4 Refining the additive uncertainty model

After all the IOTF are defined, author stresses that it is possible to recalculate the new additive uncertainty models for the subsystem  $G_j$ , that is essentially scaled by its environment.

Assuming that the subsystem  $G_j$  is somehow discretized and of reduced order (by using some MOR method) such that one has  $\tilde{G}_j$ . All the energy transfer paths are defined (as in Figure 5.16) and all the needed IOTF are calculated. If one also defines all the local IOTF connection matrices one can then examine two composite systems with either  $G_j$  or  $\tilde{G}_j$  being central to our examination (as in Figure 5.17). Due to the dissipative nature of the subsystems and their mutual interconnections, from now on, let us assume that a significant amount of the energy throughout the system was dissipated (*damped*)—in a sense, that if the response in the frequency domain is to be evaluated for any IOTF, the response will be rather smooth (without peaks of significant gain). For that matter, it makes sense not to use large order IOTF, initially obtained with unreduced subsystems. So, in the next stage of uncertainty design, reduced order IOTF will be used. For each IOTF defined with the Equations (5.1)–(5.8), one can obtain a low-order cover filters by using logarithmic-Chebyshev magnitude design. With this laid out, one has obtained two composite subsystems  $G_{j,e}$  and  $\tilde{G}_{j,e}$  (here index  $e$  will stand for *environment*). Now, as already explained in Section 5.2.3, one can proceed with the calculation of the refined uncertainty model, with refined weighting filters  $W_{j,2e}$  and  $W_{j,1e}$  now being as shown in Figure 5.18.





**Figure 5.18:** Uncertainty refinement—calculating the scaled frequency weighting filter  $W_{j,1e}$ : (a) Representation of the absolute error between the nominal scaled subsystem  $G_{j,e}$  and the reduced order scaled subsystem  $\tilde{G}_{j,e}$ . (b) Representing the absolute error as an additive uncertainty with the new refined frequency weighting filters  $W_{j,2e}$  and  $W_{j,1e}$

## CHAPTER 6

# Conclusion

### CHAPTER CONTENTS

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## 6.1 Conclusion

In this dissertation it is shown that it is possible to systematically model uncertainties arising both from spatial discretization and model order reduction. To obtain less conservative uncertainty models, a structure of the interconnected system needs to be preserved. A structure preserving algorithm that consists in partitioning the interconnected system is presented. For each subsystem of interest, partitions are made such that the input-output transfer functions (IOTFs) are calculated. IOTFs represent the energy transfer from external performance input to internal subsystem input, from internal subsystem output to external performance output, as well as from the internal subsystem output to the internal subsystem input (i.e., some feedback connection). These IOTFs are later approximated with low-order weights that essentially scale the amount of uncertainty, rendering obtained uncertainties less conservative, while keeping the overall order of the interconnected system low. This approach is shown not only to produce less conservative uncertainty models by making appropriate frequency weights for the originally obtained uncertainties, but also to introduce flexibility in both overall system modelling and uncertainty modelling. Finally, in order to validate if the models are suitable for controller synthesis, a robustness test for closed loop controller synthesis is carried out using the  $\nu$ -gap metric. The obtained  $\nu$ -gaps between newly obtained low-order robustly stable models and the nominal unreduced models, were relatively close to zero, which implies that a controller that

stabilizes the original (unreduced) interconnected system will tend to stabilize the new low-order interconnected system as well.

An important advantage of this structure preserving algorithm lies in the flexibility of the design process—a single subsystem can be easily modified and reconnected using same connections to be re-evaluated in the robustness analysis. This fact comes especially useful if a spatially invariant interconnected dynamic system is analysed or a system that has repeating subsystems with the same geometric and material properties, as for such systems memory requirements can drastically reduce. It is also possible to use other or combine different MOR methods on a subsystem level. Another advantage of the proposed algorithm lies in the flexibility of uncertainty modelling—depending on the available allowed size for the distributed robust controller to be synthesized—a trade off can be made in terms of the uncertainty conservatism versus the controller size. Usage of integral quadratic constraints framework for robustness analysis and uncertainty modelling is in itself highly attractive—besides fitting conveniently with the proposed structure-preserving scheme—it can be used to capture other types of uncertainties that can be included in the overall robustness analysis with relative ease. Some typical uncertainties that might occur and can readily be analysed, in parallel with the current analysis, include norm-bounded nonlinearities (that are used to model neglected dynamics and modelling errors), sector bounded and slope-restricted nonlinearities (that can be used for, e.g., modelling nonlinearities in material properties), passive uncertainties/nonlinearities and parametric uncertainties (concretely in the presented example this might be used to model unknown parameters in mass, stiffness and damping) [65]. Another advantage of the IQC analysis is the guaranteed robust stability and robust performance achieved by finding feasible solution of convex optimization problem.

## 6.2 Scientific contributions

Based on the presented mathematical and numerical results, discussion and finally conclusions, the hypotheses, that are once more presented below are successfully confirmed. Confirmed hypotheses are:

- Mechanical dynamical systems described with partial differential equations can be modeled as series of coupled linear time-invariant subsystems with uncertainties and by obtaining this model it is possible to describe, with sufficient accuracy, dynamic behavior crucial for synthesis of efficient control system.
- The fact that each subsystem is coupled with other subsystems may be used to obtain better uncertainty model and thus additionally improve efficiency of control system.

Besides the confirmed hypotheses, a number of the scientific contributions resulting from this dissertation can be summarized as following:

- A novel concept of multi-scale structure preservation is introduced to emphasize the importance of both global structure preservation and local structure preservation at the subsystem level.
- Errors introduced into analysis via SDMs and MOR can be modeled as unstructured linear time-invariant dynamic uncertainties.
- Uncertainty conservatism at the subsystem level can be reduced for a special class of interconnected dissipative dynamical systems.
- Preserved structure can be used to account for the dissipative dynamics of the surroundings in analysis and (uncertainty) modeling
- Systematic modeling of the uncertainty and model order reduction (MOR) at the level of a subsystem gives both modeling freedom and the ability for obtaining less conservative uncertainties on the level of a subsystem.
- $\nu$ -gap analysis can be used efficiently for verification of closed loop gain of the obtained low order robustly stable uncertain coupled dynamical system and the reference coupled dynamical system.

## 6.3 Limitations

Based on the presented, the limitations of the proposed methods and procedures are:

- The proposed method for systematical uncertainty modeling and uncertainty conservatism reduction at a level of a subsystem, although practical, is still a heuristic approach. As such it is only guaranteed to result in a sub-optimal robustly stable uncertain dynamical system.
- The local structure cannot be preserved when a first-order explicit LTI models are used. Although, arguably the local structure doesn't play a crucial role in the proposed algorithm and in a general sense, still, this problem can be partially alleviated with usage of descriptor state-space models, in which case the the only robustness analysis tool available is  $\mu$ -analysis (IQCs don't handle descriptor state-space models).
- Due to iterative nature of the proposed heuristic method and the fact that the computationally intensive IQCs are used for robustness analysis, the numerical

efficiency can be questioned. For particular problems (mainly if one only deals with unstructured uncertainties and not with mixed uncertainties) this can be alleviated by the usage of  $\mu$ -tools which were proven to give comparable results to IQCs at a fraction of calculation times.

## 6.4 Future work

A possible and attractive directions for future works include:

- With the recent development and implementation of robust control and model order reduction techniques for linear second-order dynamical systems, a clear path towards robustness analysis of proposed multi-scale structure-preserved models will be available. This could in turn yield promising results and perhaps even less conservative uncertainty models at the level of a subsystem for reduced order coupled systems. Due to local structure preservation, a lower order subsystems could be obtained with well preserved dynamics and special subsystem properties.
- A construction of an a-priori frequency weighted filters for refined uncertainty models based on energy storage functions of partitioned input-output transfer functions through the dissipation theory and integral quadratic constraints. This would be constructed from mathematical theoretic principals and a proof might be done by deriving certain LMI conditions. Solutions to LMI conditions might prove the existence of optimal stable weights for each subsystem that would a-priori guarantee the robustness of the reduced order uncertain coupled system.
- Provide mathematical proof (or carry out case study analyses, which might be rather difficult) that a current set of unstructured uncertainties can be expanded (or enriched) by modeling uncertainties as, e.g strictly dissipative uncertainty, passive uncertainty/nonlinearity or as norm bounded uncertainty/nonlinearity. For more details please see chapters 6.6 and 6.7 in [65].
- Another interested approach was also hinted in [65] in example 6. This way the weighting filters might be incorporated into performance specification using the input weighting filters through the IQCs for generalized performance.
- The heuristic nature and a large number of user decision (as shown in the flowcharts for the design procedure) gives the opportunity to consider modeling the system within the (neuro) fuzzy logic framework.

- Developing a ready to use toolbox for uncertainty modeling of reduced order spatially discretized coupled systems is something that would serve both academia as well as industry.
- A final research direction, that is most appealing to the dissertation author and in which he will hopefully embark on, relies on creating a rich enough data set through creation of a series of discretizations and reduced order models, that could be used within machine-learning framework. If an obtained data-set is to be sufficiently rich, perhaps an optimal discretization and reduced order per subsystem might be achieved. Tools that might be used, that with current authors knowledge on the topic of data-driven control come to the authors mind, include singular value decomposition, Koopman operator theory, dynamic mode decomposition and Lowener framework.

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# Curriculum Vitae

Bruno Dogančić was born on 29<sup>th</sup> of June, 1991, in Tuzla, Bosnia and Herzegovina. In 2009, he enrolled at the Faculty of Mechanical Engineering and Naval Architecture, Zagreb, where he received his bachelor's degree and master's degree with *cum laude* honor, in March 2014 and December 2015, respectively. Throughout his studies, he participated in (and won) a number of case study competitions on topics ranging from product design to smart traffic monitoring and the implementation of adaptive smart education. In March 2016, he began working in the technical department of KONČAR – SWITCHGEAR Ltd. (formerly Končar EVA - High Voltage Switchgear) as a mechanical engineer on the design and dynamical analysis of a stored-energy spring mechanism. He returned to academia in September 2016 to pursue a PhD as a Teaching and Research Assistant in the Chair for Applied Dynamics, Faculty of Mechanical Engineering and Naval Architecture, Zagreb. His research interests are focused on numerical modelling of coupled dynamical systems and control theory, including uncertainty modelling, robustness analysis, and optimal control. He published one paper as a leading author in a renowned international scientific journal indexed in SCIE (WoS CC, Q1), two papers as a leading author in reputable scientific conference proceedings indexed in CPCI (WoS CC), and co-authored a number of other conference papers.

## Most relevant publications

1. Dogančić B et al. Structure Preserving Uncertainty Modelling and Robustness Analysis for Spatially Distributed Dissipative Dynamical Systems. *Mathematics* 2022 Jun; 10 (12) :2125. DOI: 10.3390/math10122125. Available from: <https://www.mdpi.com/2227-7390/10/12/2125> [Accessed on: 2022 Jun 23]
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